Variational principle, uniqueness and reciprocity theorems in the theory of generalized thermoelastic diffusion material

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ABSTRACT
The equations of generalized thermoelastic diffusion with four relaxation times are given. The variational principle is derived. Using Laplace transforms, a uniqueness theorem for these equations is proved. Also, a reciprocity theorem is obtained.

Keywords: thermoelastic diffusion, variational principle, uniqueness theorem, reciprocity theorem
1. INTRODUCTION

In recent years increasing attention has been directed towards the generalized theory of thermoelasticity, which was found to give more realistic results than the coupled or uncoupled theories of thermoelasticity, especially when short-time effects or step temperature gradients are considered. The absence of any elasticity term in the heat conduction equation for uncoupled thermoelasticity appears to be unrealistic, since due to the mechanical loading of an elastic body, the strain so produced causes variation in the temperature field. Moreover, the parabolic type of heat conduction equation results in an infinite velocity of the thermal wave propagation, which also contradicts the actual physical phenomena. Introducing the strain-rate term in the uncoupled heat conduction equation, Biot extended the analysis to incorporate coupled thermoelasticity. In this way, although the first shortcoming was over, there remained the parabolic-type partial differential equation of heat conduction, which leads to the paradox of the infinite velocity of the thermal wave.

To take care of the paradox in Biot’s theory, Kaliski proposed a possible physical model, involving a finite velocity of heat propagation as actually required in nature. Lord and Shulman developed a theory in which they modified the Fourier’s law of heat conduction with the introduction of a thermal relaxation time parameter. The theory of generalized thermoelasticity with two relaxation times was first introduced by Müller. A more explicit version was then introduced by Green and Laws, Green and Lindsay, and independently by Suhubi. In this theory, the temperature rates are considered among the constitutive variables. This theory also predicts finite speeds of propagation as in Lord and Shulman’s theory of generalized thermoelasticity with one relaxation time. It differs from the latter in that Fourier’s law of heat conduction is not violated if the body under consideration has a center of symmetry.

Diffusion can be defined as the random walk of an assemble of particles from a region of high concentration to that of low concentration. Nowadays, there is a great deal of interest in the study of this phenomenon due to its application in geophysics and the electronic industry. In an integrated circuit fabrication, diffusion is used to introduce dopants in controlled amounts into the semiconductor substance. In particular, diffusion is used to form the base and emitter in bipolar transistors, integrated resistors, the source/drain regions in metal oxide semiconductor (MOS) transistors, and dope polysilicon gates in MOS transistors. In most applications, the concentration is calculated using what is known as Fick’s law. This is a simple law which does not take into consideration the mutual interaction between the introduced substance and the medium into which it is introduced, or the effect of temperature on this interaction. Study of the phenomenon of diffusion is used to improve the conditions of oil extraction (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits.

The phenomena of diffusion in two phase systems are well known, for example, a gas diffusing through a porous solid medium or a liquid diffusing through a solid. But until recently, the phenomena of diffusion accompanied with temperature change, that is, thermodiffusion in solids, particularly metals, was considered to be a phenomena that is independent of body deformation. It is, however, clear that such an assumption is not valid, since the processes of thermodiffusion could have a very considerable influence upon the deformation of the solid phase, and vice-versa. If the body is considered to be an ideal elastic, we may assume the validity of the linear generalized Hooke’s law, as modified by the terms representing the changes in the temperature and concentration of the diffusing phase at any given point of the body. If we also assume that during the processes of thermodiffusion, no chemical reactions take place between the two phases, then the law of conservation of mass can be taken to be valid within any given volume element. By analogy with thermoelasticity, such a phenomena of deformation will be called thermodiffusive elasticity. Of course, if the diffusion phenomena are absent, the theory will be reduced to the theory of thermoelasticity.

In the case of the thermodiffusive elasticity of a Hookean body the deformation, which is a irreversible process, occurs coupled with the irreversible processes of heat conduction and diffusion. The work in this area started with three papers by Podstrigach, Podstrigach and Pavlina, who gave the relationships between the deformation, temperature and concentration based upon the thermodynamics of irreversible processes. Podstrigach and Shevchuk presented a variational equation equivalent to the system of governing equations of a model which allows description of the interconnection between the deformation, heat and mater diffusion processes. Nowacki, in a series of papers, presented the theorem of virtual work, fundamental energy theorem, theorem of...
reciprocity of works, generalized Maxwell reciprocity relations and theorems of Somigliana and Maysel

type. Fichera proved the uniqueness, existence and the estimation of the solution in the dynamical
problems of thermodiffusion in an elastic solid. Nowacki studied dynamic problems of
thermodiffusion in solids. Herrera and Billok discussed the dual variational principles for diffusion
equations. Naerlovic´-Veljkovic´ obtained the constitutive equations for thermodiffusion in elastic,
magnetically saturated, current conducting media.

Shvets and Dasyuk proved the variational theorems of thermodiffusion in deformed solid bodies.
Gawinecki and Sierpiński proved the existence, uniqueness and regularity of the solution of the
first initial-boundary-value problem for quasistatic and dynamic equations of thermodiffusion in solid
bodies, respectively. Kubik studied the correspondence between equations of thermodiffusion and
theory of mixtures. He studied the balance equations of mass, momentum, energy and entropy. He
compared these equations and equations describing conjugate thermodiffusion flows in solids.
Wróbel investigated variational theorems for the problems of coupled thermoviscoelastic diffusion
with finite velocities of heat and mass propagation, and for the problems of thermodiffusion flows
coupled with the stress field, respectively. Gawinecki et al. proved a theorem about existence,
uniqueness and regularity of the solution to an initial-boundary value problem for a nonlinear coupled
parabolic system. They used an energy method, method of Sobolev spaces, semigroup theory and
Banach fixed point theorem to prove the theorem. Gawinecki and Szymanie proved a theorem about
global existence of the solution to the initial-value problem, for a nonlinear hyperbolic parabolic
system of coupled partial differential equation of second order, describing the process of
thermodiffusion in solid body.

Sherief, Hamza and Saleh proved the uniqueness and reciprocity theorems for the equations of
generalized thermoelastic diffusion problem, in isotropic media, on the basis of the variational
principle equations, under restrictive assumptions on the elastic coefficients. Due to the inherit
complexity of the derivation of the variational principle equations, Aouadi proved the theorem as
given by Sherief, Hamza and Saleh in the Laplace transform domain, under the assumption that the
functions of the problem are continuous and the inverse Laplace transform of each is also unique.
Aouadi derived the uniqueness and reciprocity theorems in anisotropic media, under the restriction
that the elastic, thermal conductivity and diffusion tensors are positive and definite. Aouadi
established the field equations of the linear theory of micropolar thermoelastic diffusion bodies.
Aouadi developed a theory of thermoelastic diffusion materials with voids and derived the
uniqueness, reciprocity, continuous dependence and existence theorems. Kuang derived the basic
equations and proved variational principles for generalized thermodiffusion theory in pyroelectricity.
Kumar, Kothari and Mukhopadhyay established a convolutional type variational principle and a
reciprocity theorem for the linear theory of generalized thermoelastic diffusion for isotropic elastic
solids. Ezzat and Fayik developed a new theory of thermodiffusion in elastic solids using the
methodology of fractional calculus.

Kumar and Kansal derived the constitutive relations and field equations for anisotropic generalized
thermoelastic diffusion and studied the propagation of Lamb waves in transversely isotropic
thermoelastic diffusive plate. Kumar and Kansal discussed three-dimensional free vibration analysis
of a transversely isotropic thermoelastic diffusive cylindrical panel. Kumar and Kansal studied the
propagation of plane waves and a fundamental solution in the generalized theories of thermoelastic
diffusion. Kumar and Kansal discussed the reflection and refraction phenomenon of plane waves at
the interface of an elastic solid half-space and a thermoelastic diffusive solid half-space. Kumar and
Kansal studied the propagation of plane waves in anisotropic thermoelastic diffusive medium.

2. BASIC EQUATIONS

The law of conservation of energy for an arbitrary material volume $V$ bounded by a surface $A$, at time $t$,
can be written as

$$\frac{d}{dt} \int_V \left( \frac{1}{2} \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + U \right) dV = \int_V \rho \mathbf{F}_i \dot{\mathbf{u}}_i dV + \int_A \left( \sigma_{ij} \dot{u}_j - q_j \right) n_i dA,$$

where $U$ is the internal energy per unit mass, $\rho$ is the density, $q_j$ are the components of heat flux vector
$q$, $F_i$ are components of the external forces per unit mass, $u_i$ are the components of the displacement
vector $u$, $\sigma_{ij}(=\sigma_{ji})$ are the components of the stress tensor, $n_i$ are the components of the outward unit normal vector $n$ to the surface $A$.

Using the divergence theorem and the equations of motion

$$\sigma_{ij} + \rho F_j = \rho \ddot{u}_i,$$  \hspace{1cm} (2)

the equation (1) can be written as

$$\rho \ddot{U} = \sigma_{ij} \ddot{e}_{ij} - q_{ij},$$  \hspace{1cm} (3)

where $e_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$ are the components of the strain tensor.

In what follows, we shall restrict our attention to the linear theory of homogeneous body.

The balance of entropy can be written as

$$\int \rho \dot{S}dV + \int \left( \frac{\partial \Phi}{\partial \Phi} \right) \eta_i dA - \int \left( \frac{\partial \eta_i}{\partial \Phi} \right) \phi_i dA = -q_{ij} \Phi_j - \rho_j \eta_j + \rho \Phi_j \eta_j \geq 0, \hspace{1cm} (4)$$

where $\Phi$ is a strictly positive function, $S, P$, are entropy and chemical potential per unit mass, respectively. $\eta_i$ are the components of mass diffusion flux vector $n_i$.

The right-hand side of the above equation is the entropy source

$$\mathcal{R} = -q_{ij} \Phi_j - \rho_j \eta_j + \rho \Phi_j \eta_j \geq 0. \hspace{1cm} (5)$$

In view of (5), we can write equation (4) in the form of an inequality called the Clausius Duhem inequality

$$\rho \dot{S} + q_{ij} \Phi_j - q_{ij} \Phi_j - \rho_j \eta_j + \rho \Phi_j \eta_j \geq 0. \hspace{1cm} (6)$$

The equation of conservation of mass is

$$\eta_j = -\dot{c}, \hspace{1cm} (7)$$

where $C$ is the concentration of the diffusion material in the elastic body.

Using equations (3) and (7) in equation (6), we obtain

$$\rho \dot{S} \Phi + \sigma_{ij} \dot{e}_{ij} = \rho \ddot{U} - q_{ij} \Phi_j + \rho \dot{c} + \rho \Phi_j \eta_j \geq 0, \hspace{1cm} (8)$$

If we introduce the Helmholtz free energy function $\psi$, defined by

$$\psi = U - \Phi S, \hspace{1cm} (9)$$

then from equation (8), we obtain

$$-\rho(\psi + \Phi S) + \sigma_{ij} \dot{e}_{ij} - q_{ij} \Phi_j + \rho \dot{c} + \rho \Phi_j \eta_j \geq 0. \hspace{1cm} (10)$$

We assume that

$$u_i = \beta u_i^1, T = \beta T^1, C = \beta C^1,$$

where $\beta$ is a constant small enough for squares and highest powers to be neglected and $u_i^1, T^1, C^1$ are independent of $\beta$ and $T$ is the temperature measured from the constant absolute temperature $T_0$ of the body in its reference configuration.

We denote

$$\mathcal{A} = (e_{ij}, T, T^1, \dot{T}, C, C^1, \dot{C}).$$
We consider thermoelastic diffusion solids having the constitutive equations

\[
\psi = \hat{\psi}(\mathcal{A}), \sigma_{ij} = \hat{\sigma}_{ij}(\mathcal{A}), q_i = q_i(\mathcal{A}), \quad S = \hat{S}(\mathcal{A}), P = \hat{P}(\mathcal{A}),
\]

(11)

the constitutive functionals being of class \( C^2 \) and consistent with the assumption of the linear theory. We assume that

\[
\frac{\partial \hat{\Phi}}{\partial T} \neq 0.
\]

(12)

It follows from equation (10) that

\[
\Phi = \hat{\Phi}(T, \mathcal{A}), \psi = \hat{\psi}(e_{ij}, T, \mathcal{A}, \mathcal{C}, \mathcal{C}_j, \mathcal{C}),
\]

(13)

\[
\frac{\partial \phi}{\partial T} + S \frac{\partial \Phi}{\partial T} = 0,
\]

(14)

\[
q_i = -\Phi \frac{\partial \psi}{\partial T} + \rho \eta \frac{\partial \Phi}{\partial T},
\]

(15)

\[
\left( \sigma_{ij} - \frac{\partial \psi}{\partial e_{ij}} \right) e_{ij} - \left( \frac{\partial \psi}{\partial T} + \rho \frac{\partial \Phi}{\partial T} \right) \hat{T}_i + \left( \rho - \frac{\partial \psi}{\partial \mathcal{C}} \right) \hat{\mathcal{C}} - \left( \frac{\partial \psi}{\partial \mathcal{C}} \hat{\mathcal{C}} + \frac{\partial \psi}{\partial \mathcal{C}_j} \hat{\mathcal{C}_j} \right) - \left( q_i - \frac{\rho}{\Phi} \eta \right) \frac{\partial \Phi}{\partial T} = 0,
\]

(16)

where

\[
\Psi = \rho \psi.
\]

(17)

Thus, we obtain

\[
\sigma_{ij} = \frac{\partial \Psi}{\partial e_{ij}},
\]

(18)

\[
\rho S = -\frac{\partial \psi}{\partial T} \frac{\partial \Phi}{\partial T} + \rho \frac{\partial \psi}{\partial \mathcal{C}},
\]

(19)

\[
q_i = -\Phi \frac{\partial \psi}{\partial T} \frac{\partial \Phi}{\partial T} + \rho \eta,
\]

(20)

and the inequality (16) reduces to

\[
- \left( \frac{\partial \psi}{\partial T} + \rho \frac{\partial \Phi}{\partial T} \right) \hat{T}_i + \left( \frac{\partial \psi}{\partial \mathcal{C}} \hat{\mathcal{C}} + \frac{\partial \psi}{\partial \mathcal{C}_j} \hat{\mathcal{C}_j} \right) - \left( q_i - \frac{\rho}{\Phi} \eta \right) \frac{\partial \Phi}{\partial T} = 0.
\]

(21)

Using equation (9) in the equation (3), we obtain

\[
\rho \left[ \dot{\psi} + \dot{\phi} + \dot{\phi} S \right] - \sigma_{ij} e_{ij} = -q_{ij}.
\]

(22)

In view of equations (13), (17)–(19), the energy equation (22) can be written in the form

\[
\rho \Phi \hat{S} + \left( \frac{\partial \psi}{\partial T} + \rho \frac{\partial \Phi}{\partial T} \right) \hat{T}_i + \frac{\partial \psi}{\partial \mathcal{C}} \hat{\mathcal{C}} + \frac{\partial \psi}{\partial \mathcal{C}_j} \hat{\mathcal{C}_j} = -q_{ij}.
\]

(23)

We assume that

\[
\Phi(T, 0) = T_0 + T.
\]

(24)
Assuming that the initial body is free from stress, heat flux and mass flux, in the context of linear theory, we have
\[
\Psi = \Psi_0 - kT - \nu^T \frac{1}{2} d_1 \dot{T}^2 - d_2 \dot{T}^2 - \frac{1}{2} d_3 \dot{T}^2 + \tau_1 g_i \dot{T} \dot{T}_j + \frac{1}{2} \tau_1 K_{ij} \dot{T}_j \dot{T}_j
\]
\[
+ a_{ij} e_{ij} + g_{ij} e_i \dot{T}_j + g_{ij} e_i \dot{T}_j + \frac{1}{2} b C^2 + \nu_1 \dot{C}_j
\]
\[
+ \frac{1}{2} \beta_0 \dot{T}^2 + \frac{1}{2} \dot{C}_j^2 - \tau_1 \dot{C}_j \dot{C}_j - \frac{1}{2} \tau_1 \delta_{ij} \dot{C}_i \dot{C}_j + b_{ij} e_i C + b_{ij} e_i C_j
\]
\[
+ d_{ij} e_{ij} - C_{ij} - a T C - q T \dot{C}_j - \nu_2 T \dot{C}_j - \rho T \dot{C}_j
\]
\[
- \chi T \dot{C}_j - f T \dot{C}_j - \dot{q}_i T_j \dot{C}_j - s_i T_j C + \frac{1}{2} \dot{q}_{ij} e_i e_{ij} - \nu_2 \dot{C}_j.
\]
and
\[
\Phi = T_0 + \tau_1 \dot{T} + \beta_0 \dot{T} + \frac{1}{2} \gamma_0 \dot{T}^2.
\] (25)

From equations (18)–(20) and (25), the expressions for \( \sigma_i, S, P \) and \( q_i \) are obtained as
\[
\sigma_{ij} = c_{ijm} e_{im} + a_{ij} \dot{T}_j + b_{ij} C + d_{ij} \dot{C} + b_{ij} C_j,
\] (26)
\[
\rho S = \frac{v + d_2 \dot{T} + d_3 \dot{T} + \tau_1 g_i \dot{T}_j - g_i e_j + f C + q \dot{C} + \chi C_j}{\tau_1 + \beta_0 \dot{T} + \gamma_0 \dot{T}^2},
\] (27)
\[
P = b_{ij} e_{ij} - a T - f T - s_i \dot{T}_j + b C + \nu_2 \dot{C} - \kappa C_j,
\] (28)
\[
q_i = - \frac{\Phi \dot{q}_i \dot{T}_j + \tau_1 \dot{q}_i \dot{T}_j + \tau_1 K_{ij} \dot{T}_j \dot{T}_j - \phi_i \dot{C}_j - \nu_2 \dot{C}_j}{\tau_1 + \beta_0 \dot{T} + \gamma_0 \dot{T}^2}.
\] (29)

The entropy inequality (21) implies the following restrictions on the constitutive coefficients
\[
l_i = 0, \quad v = \tau_1 k, \quad f = \tau_1 \rho, \quad \chi_i = \tau_1 \rho, \quad q = \tau_1 v, \quad v_i = 0, \quad g_{ij} = \tau_1 a_{ij}, \quad d_{2} - k \beta_0 = d_1 \tau_1,
\] (30)

and
\[
(\tau_1 d_i - h) \dot{T}^2 + 2 g_i \dot{T} \dot{T}_j + K_{ij} \dot{T}_j \dot{T}_j \geq 0,
\] (31)

where
\[
\tau_1 h = d_3 - v \gamma_0 / \tau_1.
\] (32)

Thus, we obtain the following constitutive equations
\[
\sigma_{ij} = c_{ijm} e_{im} + a_{ij} (T + \tau_1 \dot{T}_j + \tau_1 a_{ij} \dot{T}_j + b_{ij} C + d_{ij} \dot{C} + b_{ij} C_j,
\] (33)
\[
\rho S = k + d_1 \dot{T} + h \dot{T} - g_i \dot{T}_j - a_{ij} e_{ij} + a C + \nu_2 \dot{C} + \rho C_i,
\] (34)
\[
P = b_{ij} e_{ij} - a (T + \tau_1 \dot{T}_j) - s_i \dot{T}_j + b C + \nu_2 \dot{C} - \kappa C_j,
\] (35)
\[
q_i = - \tau_2 (g_i \dot{T} + K_{ij} \dot{T}_j).
\] (36)

In the context of the linear theory, the energy equation (23) reduces to
\[
\rho \Phi S = - q_{ij}.
\] (37)
If the material has centre of symmetry, then the constitutive equations (33)–(36) take the form

\[ \sigma_{ij} = c_{i j m n} e_{lm} + a_{ij}(T + \tau T) + b_{ij}(C + \tau C), \]

\[ \rho S = k + \frac{\rho C}{T_0} (T + \tau T) - a_{ij} e_{ij} + a(C + \tau C), \]

\[ \rho = b_{ij} e_{ij} + b(C + \tau C) - a(T + \tau T), \]

\[ q_i = -K_{ij} T_j, \]

where we introduce

\[ d_i = \frac{\rho C E}{T_0}, h = \frac{\rho C E}{T_0}, d_i = \tau b_{ij}, K_{ij} = K_{ij}^0 T_0, v_1 = \tau b, v_2 = \tau^0 a. \]

Analogous to equation (41) for the heat flux vector, we assume a similar equation for the mass flux vector

\[ q_i = -\alpha_i^q \rho_j, \]

In the following sections, we will use the chemical potential as a state variable instead of the concentration. The equations (38)–(40) are rewritten as:

\[ \sigma_{ij} = d_i j m e_{lm} + s_{ij}(T + \tau T) + e_{ij} \rho, \]

\[ \rho(\dot{S} + \tau \dot{S}) = k + \kappa T + \hat{A}_i T + \hat{A}_j T - s_{ij} e_{ij} + n(P + \tau^0 P), \]

\[ C + \tau^0 C = \rho P - e_{ij} e_{ij} + n(T + \tau T), (i, j, l, m = 1, 2, 3), \]

where

\[ \hat{A}_i = \frac{\rho C E}{T_0} (T_0 + \tau^0) + \frac{a^2}{b}(\tau^0 + \tau T), \quad \hat{A}_j = \frac{\rho C E}{T_0} T_0 \tau + \frac{a^2}{b} \tau^0 \tau, \quad \hat{K}_{ij} = \alpha_{ij} \tau + n b_{ij} \tau^0, \]

\[ \kappa = \frac{\rho C E}{T_0} + \frac{a^2}{b} \varphi = 1, n = \alpha \varphi, d_i j m = c_{i j m n} - \rho b_{i j} b_{l m}, s_{ij} = a_{ij} + n b_{ij}, e_{ij} = \rho b_{ij}. \]

The equation (41) with the aid of equations (37) and (45) yields

\[ K_{ij}(T + \tau T)_{ij} = T_0 [\kappa \tau + \hat{A}_i T + \hat{A}_j T - s_{ij} e_{ij} - \hat{A}_i e_{ij} + n(P + \tau^0 P)]. \]

Making use of equations (7) and (46) in the equation (43), we obtain

\[ \alpha_i^q (P + \tau^0 P)_{ij} = \rho P - e_{ij} e_{ij} + n(T + \tau T). \]

3. **VARIATIONAL PRINCIPLE**

The principle of virtual work with variation of displacements for the elastic deformable body is written as

\[ \int_V \rho[F_i - \ddot{u}_i] \delta u_i dV + \int_A h_i \delta u_i dA = \int_V \sigma_{ij} \delta u_{ij} dV. \]

On the left hand side, we have the virtual work of body forces \( F_i \), internal forces \( \rho \ddot{u}_i \), surface forces \( h_i = \sigma_{ij} n_j \), whereas on the right hand side, we have the virtual work of internal forces.

Using the symmetry of the stress tensor and the definition of the strain tensor, the equation (50) can be written in the alternative form as

\[ \int_V \rho[F_i - \ddot{u}_i] \delta u_i dV + \int_A h_i \delta u_i dA = \int_V \sigma_{ij} \delta e_{ij} dV. \]
The equation (51) with the aid of equation (44) yields

\[ \int_V \rho \left[ F_i - \dot{u}_i \right] \delta u_i dV + \int_A \gamma_i \delta u_i dA = \delta W + \int_V s_i (T + \tau_1) \delta e_{ij} dV + \int_V e_{ij} \delta e_{ij} dV, \tag{52} \]

where

\[ W = \frac{1}{2} \int_V d_{ijm} e_{ij} e_{jm} dV. \]

The equation (52) would be complete for the uncoupled problem of diffusion where the temperature \( T \) and the concentration \( C \) are known functions. When we take into account the coupling of the deformation field with the temperature and concentration, there arises the necessity of considering two additional relations characterizing the phenomena of thermal conductivity and of diffusion.

We define a vector \( J(Biot^1) \) connected with the entropy through the relation

\[ \rho S = -J_i. \tag{53} \]

Combining equations (37), (41), (45) and (53), we obtain

\[ T_0 L_{ij} \delta i + T_j = 0, \tag{54} \]

\[ -[L_{ij} + \tau^0_{ij}] k + \tau^0 k + \lambda_1 \tau + \lambda_2 \tau - s_{ij} e_{ij} - \lambda_1 \delta_{ij} + n(P + \tau^0 \delta) , \tag{55} \]

where \( L_{ij} \) is the resistivity matrix, the inverse of the thermal conductivity tensor \( K_{ij} \).

Multiplying both sides of the equation (54) by \( \delta_{ij} \) and integrating over the region of the body, we get

\[ \int_V [T_j + T_0 L_{ij}] \delta i dV = 0. \tag{56} \]

Now

\[ \int_V T_i \delta j dV = \int_V (T \delta j)_i dV - \int_V T \delta j dV. \tag{57} \]

Applying the divergence theorem defined by,

\[ \int_V (T \delta j)_i dV = \int_A (T \delta j)i dA. \tag{58} \]

in the equation (57), we obtain

\[ \int_V T_i \delta j dV = \int_A (T \delta j)i dA - \int_V T \delta j dV. \tag{59} \]

Equation (56) with the aid of equation (59) yields

\[ \int_A (T \delta j)_i dA - \int_V T \delta j dV + T_0 \int_V L_{ij} \delta j dV = 0. \tag{60} \]

Substituting the value of \( J_{ij} \) from equation (55) in the equation (60) gives the second variational equation

\[ \beta_1 + \tau^0 \beta_1 + \lambda_1 \int_V T \delta T dV + \lambda_2 \int_V T \delta T dV - \int_V s_{ij} T \delta e_{ij} dV - \int_V \lambda_1 \delta \delta e_{ij} dV + n \int_V T \delta p dV + n \tau_0 \int_V T \delta p dV + \delta E + H^* + \tau^0 H^* = 0, \tag{61} \]
where
\[
\beta_2^* = \int_A (T \delta_j) n_j dA, \quad H^* = \frac{T_o}{2} \int_V L_{ij} \delta_j dV, \quad \delta H^* = T_o \int_V L_{ij} \delta_j dV, \quad E = \frac{\kappa}{2} \int_V T^2 dV, \quad \delta E = \kappa \int_V T \delta T dV.
\]

In order to obtain the last of the variational equations, we now introduce the vector function \( \mathbf{N} \) defined as follows
\[
C = -N_{ij}.
\]  
(62)

Combining equations (7), (43), (46) and (62), we obtain
\[
\alpha_0 \dot{N}_i + P_j = 0,
\]  
(63)

\[-[N_{ij} + \tau^2 N_{ij}] = \rho P - \varepsilon e_i e_j + n(T + \tau_1 
\]  
(64)

where \( \alpha_0 \) is the inverse of the diffusion tensor \( \alpha_0^* \).

Multiplying equation (63) by \( \delta N_i \) and integrating over the region of the body, we obtain
\[
\int_V [\alpha_0 \dot{N}_i + P_j] \delta N_i dV = 0,
\]  
(65)

Consider
\[
\int_V P_j \delta N_i dV = \int_V (P \delta N_i)_j dV - \int_V P \delta N_{ij} dV.
\]  
(66)

We know that
\[
\int_V (P \delta N_i)_j dV = \int_A (P \delta N_i) n_j dA.
\]  
(67)

Substituting the value of \( \int_V (P \delta N_i)_j dV \) from equation (67) in the equation (66) gives
\[
\int_V P_j \delta N_i dV = \int_A (P \delta N_i) n_j dA - \int_V P \delta N_{ij} dV.
\]  
(68)

Making use of equation (68) in the equation (65) yields
\[
\int_A (P \delta N_i) n_j dA - \int_V P \delta N_{ij} dV + \int_V \alpha_0 \dot{N}_i \delta N_i dV = 0.
\]  
(69)

Using equation (64) in the equation (69), we obtain the third variational equation
\[
\beta_2^* + \tau^2 \dot{\beta}_2^* = \int_V e_i P \delta e_j dV + \int_V P \delta T dV + \int_V \delta (\delta + G^* + \tau^2 G^*) = 0,
\]  
(70)

where
\[
\delta \tilde{F} = \frac{\rho}{2} \int_V P^2 dV, \quad \delta \tilde{F} = \rho \int_V P \delta P dV.
\]
Eliminating integrals $\int_V s_i T \delta e_i dV$ and $\int_V e_i P \delta e_i dV$ from equations (52), (63) and (70), we obtain the final variational principle in the following form

$$
\delta (W + E + H^* + \tilde{F} + G^* + \tau (H^* + \tilde{G}^*)) + \int_V PT dV = \int_V \rho [\tilde{F}_i - \tilde{u}_i] \delta u_i dV
$$

$$
+ \int_A h_i \delta u_i dA - \beta_i^* - \beta_i^* - \tau (\tilde{\beta}_i^* + \tilde{\beta}_i^*) - \tilde{A}_i \int_V T \delta T dV - \tilde{A}_i \int_V T \delta T dV - \tau, \int_V s_i T \delta e_i dV + \int_V \tilde{A}_i T \delta e_i dV - n \tau \int_V P \delta T dV - n \tau \int_V P \delta T dV.
$$

(71)

On the right-hand side of equation (71), we find all the causes, the mass forces, inertial forces, the surface forces, the heating and the chemical potential on the surface bounding the body.

4. **UNIQUENESS THEOREM**

Assuming a linear anisotropic thermoelastic diffusion material occupies a regular region $V$ with boundary surface $A$ in the three-dimensional space, there is only system of functions

$$
u_i (x, t), T (x, t), P (x, t) \text{ of class } C^{(2)} \text{, and } \sigma_i (x, t), e_i \text{ of class } C^{(1)} \text{in the point } \tilde{P} e V + A
$$

having coordinates $x = (x_1, x_2, x_3)$ at $t \geq 0$, satisfy the equation (44) and relation $e_i = \frac{1}{2} (u_i, u_i)$ for $x e V + A, t \geq 0$ and equations (2), (48) and (49) for $x e V, t > 0$ with boundary conditions

$$
u_i = \tilde{\nu}_i^{(1)} (x, t), T = \tilde{T}^{(1)} (x, t) \text{ and } P = \tilde{P}^{(1)} (x, t) x e A, t > 0,
$$

(72)

and the initial conditions

$$
u_i = \tilde{\nu}_i^{(2)} (x, o), \tilde{T}^{(2)} (x, o), T = \tilde{T}^{(2)} (x, o),
$$

(73)

$$
\tilde{\nu}_i = \tilde{\nu}_i^{(1)} (x, o), P = \tilde{P}^{(1)} (x, o) x e V.
$$

We assume that the material parameters satisfy the inequalities

$$
T_0 > 0, \tau_0 > 0, \rho > 0, \tau^0 > 0, C_E > 0, \tau_1 > 0, a > 0, b > 0, \tau_1 > 0,
$$

(74)

d_{ijm}, K_j, \alpha_{ij}^m \text{ are positive definite.}

**Proof.** Let $\tilde{\nu}_i^{(1)}, T^{(1)}, P^{(1)}, \ldots$ and $\tilde{\nu}_i^{(2)}, T^{(2)}, P^{(2)}, \ldots$ be two solutions sets of equations (2), (44), (48) and (49) with the same body forces, the same relaxation functions, the same boundary conditions (72) and the same initial conditions (73). Consider the difference functions

$$
\tilde{\nu}_i = \tilde{\nu}_i^{(1)} - \tilde{\nu}_i^{(2)}, \tilde{T} = T^{(1)} - T^{(2)}, \text{ and } \tilde{P} = P^{(1)} - P^{(2)}, \ldots
$$

(75)

which satisfy equations (2), (44), (48) and (49). Thus equations (2), (44), (48) and (49) for the difference functions become

$$
\tilde{\sigma}_{ij} = \tilde{\rho} \tilde{\nu}_i,
$$

(76)

$$
\sigma_i = d_{ijm} \tilde{\nu}_m + s_i (\tilde{T} + \tau \tilde{T}) + e_i \tilde{P},
$$

(77)

$$
K_i (\tilde{T} + \tau \tilde{T}), \tilde{\nu}_i = T_0 (x \tilde{T} + A_1 \tilde{T} + \tau \tilde{T} - s_i \tilde{\nu}_i - \tilde{A}_i \tilde{\nu}_i + n (\tilde{P} + \tau \tilde{P}),
$$

(78)

$$
\alpha_{ij}^m (\tilde{P} + \tau \tilde{P}), \tilde{\nu}_i = \tilde{\nu}_i, \tilde{\nu}_i = n (\tilde{T} + \tau \tilde{T}),
$$

(79)

The difference functions (75) also satisfy the homogeneous boundary and initial conditions. Thus,

$$
\tilde{\nu}_i (x, t) = 0, \tilde{T} (x, t) = 0 \text{ and } \tilde{P} (x, t) = 0 \text{ x e A, t > 0},
$$

(80)
\( \ddot{u}(x, t) = 0, \quad \dot{u}(x, t) = 0, \quad \tau(x, t) = 0, \quad \dot{\tau}(x, t) = 0, \quad \ddot{\tau}(x, t) = 0 \) and \( \dot{\tau}(x, t) = 0 \) \( x \neq V \). \hspace{1cm} (81)

Applying the Laplace transform defined by

\[
\tilde{f}(x, s) = \tilde{E}(f(x, t)) = \int_0^\infty f(x, t)e^{-st}dt.
\]

(82)

on the equations (72) and (76)–(79), and omitting bars for simplicity, we obtain

\[
\tilde{\sigma}_{ij} = \rho s^2 \tilde{u}_i,
\]

(83)

\[
\tilde{\sigma}_i = d_{ij\alpha\beta} \tilde{e}_{\alpha\beta} + s_{ij}(1 + \tau_s) \tilde{\tau} + \epsilon_{ij} \tilde{\rho}.
\]

(84)

\[
(1 + \tau^s)K_{ij\tilde{\sigma}} = sT_0[(\kappa + \lambda_s + \lambda_2 s^2) \tilde{\tau} - (s_{ij} + \lambda_3 s) \tilde{\rho} + n(1 + \tau_s) \tilde{\rho}].
\]

(85)

\[
(1 + \tau^s)K_{ij\tilde{\rho}} = s[\rho \tilde{\sigma} - \epsilon_{ij} \tilde{\rho} + [(1 + \tau_s) \tilde{\tau}],
\]

(86)

\[\tilde{u}(x, s) = 0, \quad \tilde{\tau}(x, s) = 0 \quad \text{and} \quad \tilde{\rho}(x, s) = 0 \quad x \neq V. \hspace{1cm} (87)\]

Consider the integral

\[
\int_V (\tilde{\sigma}_{ij} \tilde{u}_j) dV = \int_V (\tilde{\sigma}_{ij} \tilde{u}_j) dV - \int_A (\tilde{\sigma}_{ij} \tilde{u}_j) dA = 0.
\]

(88)

Using the divergence theorem and taking into consideration (87), we obtain

\[
\int_V (\tilde{\sigma}_{ij} \tilde{u}_j) dV = \int_A (\tilde{\sigma}_{ij} \tilde{u}_j) dA = 0.
\]

(89)

Therefore, equation (88) with the aid of equation (89) takes the form

\[
\int_V \tilde{\sigma}_{ij} \tilde{u}_j dV + \int_V \tilde{\sigma}_{ij} \tilde{\rho}_j dV = 0.
\]

(90)

Using equations (83) and (84) in the equation (90), we obtain

\[
\int_V [d_{ij\alpha\beta} \tilde{e}_{\alpha\beta} + s_{ij}(1 + \tau_s) \tilde{\tau} + \epsilon_{ij} \tilde{\rho} + \rho s^2 \tilde{u}_i \tilde{u}_j] dV = 0.
\]

(91)

Now,

\[
\tilde{T}_{ij} = (\tilde{T}_{ij}) - \tilde{T}_{ij}, \quad \text{and} \quad \int_V (\tilde{T}_{ij}) dV = \int_A (\tilde{T}_{ij}) dA = 0.
\]

(92)

Therefore, we obtain

\[
\int_V \tilde{\tau}(K_{ij} \tilde{\sigma}) dV = \int_V K_{ij}(\tilde{T}_{ij}) dV - \int_V K_{ij} \tilde{T}_{ij} dV = - \int_V K_{ij} \tilde{T}_{ij} dV.
\]

(93)

The equation (93) with the aid of equation (85) yields

\[
sT_0[s_{ij} + \lambda_3 s] \int_V \tilde{\tau} dV = (1 + \tau^s) \int_V K_{ij} \tilde{T}_{ij} dV +
\]

\[
sT_0[(\kappa + \lambda_s + \lambda_2 s) + \lambda_2 s^2] \int_V \tilde{\tau} dV + nsT_0(1 + \tau^s) \int_V \tilde{\rho} dV.
\]

(94)

Analogous to equation (92), we have

\[
\tilde{\rho}_{ij} = (\tilde{\rho}_{ij})_{ij} - \tilde{\rho}_{ij}, \quad \text{and} \quad \int_V (\tilde{\rho}_{ij}) dV = \int_A (\tilde{\rho}_{ij}) dA = 0.
\]

(95)
Thus, we obtain
\[
\int_V \rho(\alpha_i^j P_{\bar{j}})dV = \int_V \alpha_i^j (\rho P_{\bar{j}})dV - \int_V \alpha_i^j P_{\bar{j}}dV = - \int_V \alpha_i^j P_{\bar{j}}dV. \tag{96}
\]

Using equation (86) in the equation (96), we obtain
\[
s \int_V e_i \tilde{P}dV = (1 + \tau^{ij}) \int_V \alpha_i^j \tilde{P} \tilde{P}_dV + \rho s \int_V \tilde{P}^2dV + ns(1 + \tau^{ij}) \int_V \tilde{P}dV. \tag{97}
\]

The equation (91) with the aid of equations (94) and (97) gives
\[
s[s_i + \tilde{A}_i] \int_V d_{ij} \bar{e}_i \bar{e}_j dV + \frac{(1 + \tau^{ij})(1 + \tau^{ij})}{\tilde{T}_0} \int_V s_{ij} \tilde{T}_j \tilde{T}_j dV + (1 + \tau^{ij})s \int_V s_{ij}[(\kappa + \tilde{A}_i + \tilde{A}_2s^2)\tilde{T}^2 + n(1 + \tau^0s)\tilde{T}]dV + (1 + \tau^{ij})(1 + \tau^{ij}) \int_V \alpha_i^j \tilde{P}_i \tilde{P}_j dV + \rho s \int_V \tilde{P}^2dV + ns(1 + \tau^{ij}) \int_V \tilde{P}dV + \rho s^3 \tilde{u}_i \tilde{u}_idV = 0. \tag{98}
\]

Substituting the values of $s_{ij}$ and $\tilde{A}_i$ from equation (47) in the equation (98), and comparing the terms containing $a_{ij}$ and $b_{ij}$ on both sides, we obtain
\[
s(1 + \tau^{ij}) \int_V d_{ij} \bar{e}_i \bar{e}_j dV + \frac{(1 + \tau^{ij})(1 + \tau^{ij})}{\tilde{T}_0} \int_V K_{ij} \tilde{T}_j \tilde{T}_j dV + (1 + \tau^{ij})s \int_V [(\kappa + \tilde{A}_i + \tilde{A}_2s^2)\tilde{T}^2 + n(1 + \tau^0s)\tilde{T}]dV + (1 + \tau^{ij})(1 + \tau^{ij}) \int_V \alpha_i^j \tilde{P}_i \tilde{P}_j dV + \rho s \int_V \tilde{P}^2dV + ns(1 + \tau^{ij}) \int_V \tilde{P}dV + \rho s^3 \tilde{u}_i \tilde{u}_idV = 0. \tag{99}
\]

and
\[
s(1 + \tau^0s) \int_V d_{ij} \bar{e}_i \bar{e}_j dV + \frac{(1 + \tau^{ij})(1 + \tau^{ij})}{\tilde{T}_0} \int_V K_{ij} \tilde{T}_j \tilde{T}_j dV + (1 + \tau^{ij})s \int_V [(\kappa + \tilde{A}_i + \tilde{A}_2s^2)\tilde{T}^2 + n(1 + \tau^0s)\tilde{T}]dV + (1 + \tau^0)(1 + \tau^{ij}) \int_V \alpha_i^j \tilde{P}_i \tilde{P}_j dV + \rho s \int_V \tilde{P}^2dV + ns(1 + \tau^{ij}) \int_V \tilde{P}dV + \rho s^3 \tilde{u}_i \tilde{u}_idV = 0. \tag{100}
\]

The equation (99) can be rewritten as
\[
\int_V \left[ \hat{m}_1 d_{ij} \bar{e}_i \bar{e}_j + \hat{m}_2 K_{ij} \tilde{T}_j + \hat{m}_3 \tilde{T}^2 \right] + (\hat{m}_4 + \hat{m}_7) \tilde{P} + \hat{m}_5 \alpha_i^j \tilde{P}_i + \hat{m}_6 \tilde{P}^2 + \rho s^3 \tilde{u}_i \tilde{u}_i dV = 0, \tag{101}
\]
where
\[
\hat{m}_1 = s(1 + \tau^s), \hat{m}_2 = \frac{(1 + \tau_1 s)(1 + \tau^s)}{\tau}, \hat{m}_3 = (1 + \tau_1 s) s (\kappa + \hat{A}_i s + \hat{A}_2 s^2),
\]
\[
\hat{m}_4 = n s(1 + \tau_1 s)(1 + \tau^s), \hat{m}_5 = (1 + \tau_5)^2, \hat{m}_6 = \rho s(1 + \tau^s), \hat{m}_7 = n s(1 + \tau_1 s)(1 + \tau^s).
\]

Using positive definiteness conditions (74) for non-zero tensors \( d_{ijm} \hat{e}_{ij} \hat{e}_{ij} \geq \hat{d}_{ij} \hat{e}_{ij}, \ K_{ij} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} \geq \hat{k} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij}, \ \alpha_{ij} \hat{P}_{ij} \hat{P}_{ij} \geq \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} \) for some positive constants \( \hat{d}, \hat{k} \) and \( \hat{\alpha} \), inequality (101) takes the form

\[
\int [\hat{m}_1 \hat{d}_{ij} \hat{e}_{ij} + \hat{m}_2 \hat{k} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_3 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_4 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_5 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_6 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \rho s^3 \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} dV \leq 0. \tag{102}
\]

Now
\[
\hat{m}_1 > 0, \hat{m}_2 > 0, \hat{m}_4 > 0, \rho > 0. \tag{103}
\]

We also note that the expression \( \int \int [\hat{m}_3 \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_4 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + (\hat{m}_4 + \hat{m}_7) \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_6 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \rho s^3 \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} dV \) occurring in the expression (102) is always positive, since, by the laws of thermodynamics
\[
o < \hat{m}_4 \hat{\rho} < \hat{m}_7 \hat{\rho}.
\tag{104}
\]

Thus, inequality (102) becomes

\[
\int [\hat{m}_1 \hat{d}_{ij} \hat{e}_{ij} + \hat{m}_2 \hat{k} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_3 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_4 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_5 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_6 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \rho s^3 \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} dV \leq 0. \tag{105}
\]

The integrated function in equation (105) is the sum of squares with positive coefficients, thus we conclude that the only possibility for the integral is to equal zero, that is,

\[
\int [\hat{m}_1 \hat{d}_{ij} \hat{e}_{ij} + \hat{m}_2 \hat{k} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_3 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_4 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \hat{m}_5 \hat{\alpha} \hat{P}_{ij} \hat{P}_{ij} + \hat{m}_6 \hat{\rho} \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} + \rho s^3 \hat{\epsilon}_{ij} \hat{\epsilon}_{ij} dV = 0. \tag{106}
\]

From equation (106), it is concluded that
\[
\hat{u}_i = 0, \ \hat{T}_i = 0, \ \hat{P}_i = 0, \ \hat{e}_{ij} = 0. \tag{107}
\]

Equations (85) and (86) with the aid of equation (107) yield
\[
(\kappa + \hat{A}_i s + \hat{A}_2 s^2) \hat{T} + n(1 + \tau^s) \hat{P} = 0, \tag{108}
\]
\[
n(1 + \tau_1 s) \hat{T} + \rho \hat{P} = 0. \tag{109}
\]

The system of equations (108) and (109) has a trivial solution, because
\[
\begin{vmatrix}
\kappa + \hat{A}_i s + \hat{A}_2 s^2 & n(1 + \tau^s) \\
(1 + \tau_1 s) & \rho
\end{vmatrix} \neq 0. \tag{110}
\]

Therefore, we obtain
\[
\hat{T} = 0, \ \hat{P} = 0. \tag{111}
\]

Using equations (107) and (111) in the equation (84), we obtain
\[
\hat{d}_{ij} = 0. \tag{112}
\]

Thus, equations (107), (111) and (112) yield
\[
\hat{u}_i = 0, \ \hat{T} = 0, \ \hat{P} = 0, \ \hat{e}_{ij} = 0, \ \hat{d}_{ij} = 0. \tag{113}
\]

Similarly, from equation (100), we can obtain the same results as given in equation (113).
Therefore, the Laplace transforms of all the difference functions (75) are zeros, and, since they are continuous functions, the inverse Laplace transform of each is unique. This proves the uniqueness of the solution of the system of equations (2), (44), (48) and (49).

4. RECIPROCITY THEOREM
We shall consider a homogeneous anisotropic generalized thermoelastic diffusion body occupying the region \( V \) and bounded by the surface \( A \). We assume that the stresses \( \sigma_i \) and the strains \( e_i \) are continuous together with their first derivatives, whereas the displacements \( u_i \), temperature \( T \), concentration \( C \) and the chemical potential \( \rho \) are continuous and have continuous derivatives up to the second order, for \( x \in V + A, \ t > 0 \).

The components of surface traction, the normal component of heat flux and the normal component of chemical flux at regular points of \( \partial V \), are given by

\[
h_i = \sigma_i n_i, \quad q = K_i T n_i, \quad p = \alpha^*_i \rho n_i,
\]

respectively.

To the system of field equation, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

\[
u_i(x, t) = \dot{u}_i(x, t), \quad T(x, t) = \eta(x, t), \quad P(x, t) = \varphi(x, t),
\]

for all \( x \in A, \ t > 0 \).

and the homogeneous initial conditions

\[
u_i(x, 0) = \dot{u}_i(x, 0) = 0, \quad T(x, 0) = \eta(x, 0) = 0, \quad P(x, 0) = \varphi(x, 0) = 0,
\]

for all \( x \in V, \ t = 0 \).

We derive the static reciprocity relationship for a generalized thermoelastic diffusion bounded body \( V \), which satisfies equations (2), (44), (48) and (49), the boundary conditions (115) and the homogeneous initial conditions (116), and are subjected to the action of body forces \( F_i(x, t) \), surface traction \( h_i(x, t) \), the heat flux \( q_i(x, t) \) and the chemical flux \( \rho(x, t) \).

Applying the Laplace transform defined by equation (82) on equations (2), (44), (48) and (49) and omitting the bars for simplicity, we obtain

\[
\sigma_{ij} + \rho F_i = \rho s^2 u_i,
\]

\[
\sigma_{ij} = d_{ij} u_{i0} + s_{ij}(1 + \tau_s)T + e_{ij} P,
\]

\[
(1 + \tau^s)K_i T_{ij} = sT_0[(\kappa + \bar{A}_s + \bar{A}_2 s^2)T - (s_{ij} + \bar{A}_s e_{ij}) + n(1 + \tau^s)P],
\]

\[
(1 + \tau^s)\alpha^*_i \rho_{ij} = s[\rho^P e_{ij} + n(1 + \tau_s)T].
\]

We now consider two problems where applied body forces, chemical potential and the surface temperature are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have \( u^{(1)}, \ e^{(1)}, \ d^{(1)}, \ T^{(1)}, \ \rho^{(1)}, \ldots \) for the first problem and \( u^{(2)}, \ e^{(2)}, \ d^{(2)}, \ T^{(2)}, \ \rho^{(2)}, \ldots \) for the second problem. Each set of variables satisfies the equations (2), (44), (48) and (49).

Using the assumption \( \sigma_{ij} = \sigma_{ji} \), we obtain

\[
\int_V \alpha^{(1)}_{ij} e^{(2)}_{ij} dV = \int_V \alpha^{(2)}_{ij} e^{(1)}_{ij} dV = \int_v (\alpha^{(1)}_{ij} u^{(2)}_{ij}) dV - \int_v (\alpha^{(2)}_{ij} u^{(1)}_{ij}) dV.
\]
Using the divergence theorem in the first term of the right hand side of equation (121) yields

\[
\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \int_A (\sigma_{ij}^{(1)} u_i^{(2)}) n_i dA - \int_V \sigma_{ij}^{(1)} u_i^{(2)} dV. \tag{122}
\]

Equation (122) with the use of equations (114) and (117) yields

\[
\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV = \int_A \left( h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)} \right) dA + \rho \int_V F_i^{(1)} u_i^{(2)} dV. \tag{123}
\]

A similar expression is obtained for the integral \( \int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} dV \), which together with equation (123), it follows that

\[
\int_V \left[ \sigma_{ij}^{(3)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)} \right] dV = \int_A \left[ h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)} \right] dA + \rho \int_V F_i^{(1)} u_i^{(2)} dV. \tag{124}
\]

Now multiplying equation (118) by \( e_{ij}^{(2)} \) and \( e_{ij}^{(1)} \) for the first and second problems respectively, subtracting and integrating over the region \( V \), we obtain

\[
\int_V \left[ \sigma_{ij}^{(1)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)} \right] dV = \int_V d_{ijlm} (e_{im}^{(1)} e_{lj}^{(2)} - e_{im}^{(2)} e_{lj}^{(1)}) dV
\]

\[+ \int_V s_{ij}(1 + \tau s) (T^{(1)} e_{ij}^{(2)} - T^{(2)} e_{ij}^{(1)}) dV + \int_V e_{ij} (p^{(1)} e_{ij}^{(2)} - p^{(2)} e_{ij}^{(1)}) dV. \tag{125}\]

Using the symmetry properties of \( d_{ijlm} \), we obtain

\[
\int_V \left[ \sigma_{ij}^{(3)} e_{ij}^{(2)} - \sigma_{ij}^{(2)} e_{ij}^{(1)} \right] dV = \int_V s_{ij}(1 + \tau s) (T^{(1)} e_{ij}^{(2)} - T^{(2)} e_{ij}^{(1)}) dV + \int_V e_{ij} (p^{(1)} e_{ij}^{(2)} - p^{(2)} e_{ij}^{(1)}) dV. \tag{126}\]

From equations (124) and (125), we get the first part of the reciprocity theorem

\[
\int_A \left[ h_i^{(1)} u_i^{(2)} - h_i^{(2)} u_i^{(1)} \right] dA + \rho \int_V \left[ F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)} \right] dV
\]

\[= \int_V s_{ij}(1 + \tau s) (T^{(1)} e_{ij}^{(2)} - T^{(2)} e_{ij}^{(1)}) dV + \int_V e_{ij} (p^{(1)} e_{ij}^{(2)} - p^{(2)} e_{ij}^{(1)}) dV, \tag{126}\]

which contains the mechanical causes of motion \( F_i \) and \( h_i \).

To derive the second part, multiplying equation (119) by \( T^{(1)} \) and \( T^{(2)} \) for the first and second problems respectively, subtracting and integrating over \( V \), we get

\[
(1 + \tau^s) \int_V ((K_{ij}^{(1)} T_i^{(1)}) T_j^{(2)} - (K_{ij}^{(2)} T_i^{(2)}) T_j^{(1)}) dV = -(s_{ij} + \Lambda_{ij}^s) s T_{ij} \int_V (e_{ij}^{(1)} T^{(2)} - e_{ij}^{(2)} T^{(1)}) dV
\]

\[+ ns(1 + \tau^s) s T_{ij} \int_V (p^{(1)} T^{(2)} - p^{(2)} T^{(1)}) dV. \tag{127}\]

Now

\[
(K_{ij}^{(1)} T_i^{(1)}) T_j^{(2)} = (K_{ij}^{(1)} T_i^{(1)}) T_j^{(2)} - K_{ij}^{(2)} T_i^{(2)} T_j^{(2)}, \quad \text{and} \quad (K_{ij}^{(2)} T_i^{(2)}) T_j^{(1)} = (K_{ij}^{(2)} T_i^{(2)}) T_j^{(1)} - K_{ij}^{(2)} T_i^{(2)} T_j^{(2)}. \tag{128}\]
Equation (127) with the help of equations (114), (115), (128) and the divergence theorem, can be written as

\[
(1 + \tau^5) \int_A (q^{(1)} - q^{(2)}) dA = -(s_y + \Lambda_y s) T_o \int_V (\epsilon^{(1)} T^{(2)} - \epsilon^{(2)} T^{(3)}) dV \\
+ ns(1 + \tau^5) T_o \int_V n (P^{(1)} T^{(2)} - P^{(2)} T^{(3)}) dV. 
\]  

(129)

To derive the third part, multiplying equation (120) by \( P^{(2)} \) and \( P^{(1)} \) for the first and second problems respectively, subtracting and integrating over \( V \), we get

\[
(1 + \tau^5) \int_V (\alpha^* \rho^{(2)} - (\alpha^*)^* \rho^{(2)} - (\alpha^* \rho^{(2)}) - (\alpha^*)^* \rho^{(2)}) dV = -(s_y + \Lambda_y s) \int_V (\epsilon^{(1)} p^{(2)} - \epsilon^{(2)} p^{(3)}) dV \\
+ ns(1 + \tau^5) \int_V (P^{(2)} T^{(1)} - P^{(1)} T^{(2)}) dV. 
\]  

(130)

Consider

\[
(\alpha^* \rho^{(2)}) = (\alpha^* \rho^{(1)} \rho^{(2)}) - (\alpha^*)^* \rho^{(2)} \rho^{(2)}, \text{ and} \\
(\alpha^* \rho^{(2)}) = (\alpha^* \rho^{(1)} \rho^{(2)}) - (\alpha^*)^* \rho^{(2)} \rho^{(2)}. 
\]  

(131)

Equation (130) with the aid of equations (114), (115), (131) and the divergence theorem yields

\[
(1 + \tau^5) \int_A (P^{(1)} s^{(2)} - P^{(2)} s^{(1)}) dA = -(s_y + \Lambda_y s) \int_V (\epsilon^{(1)} p^{(2)} - \epsilon^{(2)} p^{(3)}) dV \\
+ ns(1 + \tau^5) \int_V (P^{(2)} T^{(1)} - P^{(1)} T^{(2)}) dV. 
\]  

(132)

Eliminating the integrals \( \int_V (\epsilon^{(2)} T^{(1)} - \epsilon^{(1)} T^{(2)}) dV \) and \( \int_V (\epsilon^{(2)} P^{(1)} - \epsilon^{(2)} P^{(2)}) dV \) from equations (126), (129) and (132), we obtain

\[
sT_o (s_y + \Lambda_y s) \int_A \left( h^{(1)} \rho^{(1)} - h^{(2)} \rho^{(2)} \right) dA + \rho \int_V \left( \epsilon^{(1)} \rho^{(2)} - \epsilon^{(2)} \rho^{(3)} \right) dV = -s_y(1 + \tau^5) \int_A (q^{(1)} \eta^{(2)} - q^{(2)} \eta^{(1)}) dA \\
T_o (s_y + \Lambda_y s) \int_A \left( P^{(1)} s^{(2)} - P^{(2)} s^{(1)} \right) dA = -nsT_o (1 + \tau^5) (\tau^1 - \tau^0) \int_V (P^{(1)} \eta^{(2)} - P^{(2)} \eta^{(2)}) dV = 0. 
\]  

(133)

This is the general reciprocity theorem in the Laplace transform domain. Applying the inverse Laplace transform defined by

\[
\mathcal{L}^{-1}[F(s)G(s)] = \int_0^s f(t - \xi)g(\xi) d\xi = \int_0^s g(t - \xi)f(\xi) d\xi, 
\]  

(134)
and the symbolic notations

\[
\begin{align*}
\hat{p}_1(f) &= 1 + \tau \frac{\partial f(x, \xi)}{\partial \xi}, \\
\hat{p}_2(f) &= 1 + \tau^2 \frac{\partial f(x, \xi)}{\partial \xi^2}, \\
\hat{p}_3(f) &= 1 + \tau^3 \frac{\partial^3 f(x, \xi)}{\partial \xi^3}, \\
\hat{p}_4(f) &= 1 + (\tau + \tau^2) \frac{\partial f(x, \xi)}{\partial \xi} + \tau^3 \frac{\partial^2 f(x, \xi)}{\partial \xi^2} + \tau^4 \frac{\partial^3 f(x, \xi)}{\partial \xi^3}, \\
\hat{p}_5(f) &= 1 + (\tau^3 + \tau^2) \frac{\partial f(x, \xi)}{\partial \xi} + \tau^4 \frac{\partial^2 f(x, \xi)}{\partial \xi^2} + \tau^5 \frac{\partial^3 f(x, \xi)}{\partial \xi^3}, \\
\hat{p}_6(f) &= 1 + 2\tau^2 \frac{\partial f(x, \xi)}{\partial \xi} + (\tau^2)^2 \frac{\partial^2 f(x, \xi)}{\partial \xi^2}.
\end{align*}
\]

(135)

on the equations (126), (129) and (132) respectively yield the first, second and third parts of the reciprocity theorem in the final form

\[
\begin{align*}
\int_A \int_0^t h^{(1)}(x, t - \xi)u^{(2)}(x, \xi) d\xi dA + \rho \int_V \int_0^t F^{(1)}(x, t - \xi)u^{(2)}(x, \xi) d\xi dV \\
- \int_V \int_0^t s_j \gamma^{(1)}(x, t - \xi)\hat{p}_1(e^{(2)})(x, \xi) d\xi dV \\
- \int_V \int_0^t e_j \rho^{(1)}(x, t - \xi)e^{(2)}(x, \xi) d\xi dV = S_{21}^{12},
\end{align*}
\]

(136)

\[
\begin{align*}
\int_A \int_0^t q^{(1)}(x, t - \xi)\hat{p}_2(\eta^{(2)})(x, \xi) d\xi dA - T_0 \int_V \int_0^t a_i \gamma^{(1)}(x, t - \xi) \frac{\partial \hat{p}_2}{\partial \xi} (e^{(2)})(x, \xi) d\xi dV \\
- nT_0 \int_V \int_0^t b_j \gamma^{(1)}(x, t - \xi) \frac{\partial \hat{p}_3}{\partial \xi} (\rho^{(2)})(x, \xi) d\xi dV \\
+ nT_0 \int_V \int_0^t \gamma^{(1)}(x, t - \xi) \frac{\partial \hat{p}_4}{\partial \xi} (\gamma^{(2)})(x, \xi) d\xi dV = S_{21}^{12},
\end{align*}
\]

(137)

and

\[
\begin{align*}
\int_A \int_0^t p^{(1)}(x, t - \xi)\hat{p}_2(s^{(2)})(x, \xi) d\xi dA - \int_V \int_0^t e_j \rho^{(1)}(x, t - \xi) \frac{\partial \hat{p}_2}{\partial \xi} (s^{(2)})(x, \xi) d\xi dV \\
- n \int_V \int_0^t \gamma^{(1)}(x, t - \xi) \frac{\partial \hat{p}_3}{\partial \xi} (\rho^{(2)})(x, \xi) d\xi dV = S_{21}^{12},
\end{align*}
\]

(138)

Here $S_{21}^{12}$ indicates the same expression as on the left-hand side, except that superscripts (1) and (2) are interchanged.

Finally, equation (133) with the aid of equations (134) and (135), gives the general reciprocity theorem in the final form

\[
\begin{align*}
T_0 \int_A \int_0^t h^{(1)}(x, t - \xi)[a_i - \frac{\partial \hat{p}_2}{\partial \xi} (u^{(2)})(x, \xi)] d\xi dA \\
+ \rho T_0 \int_V \int_0^t F^{(1)}(x, t - \xi)[a_i - \frac{\partial \hat{p}_2}{\partial \xi} (u^{(2)})(x, \xi)] d\xi dV \\
- \int_A \int_0^t s_j \gamma^{(1)}(x, t - \xi)\hat{p}_4(\eta^{(2)})(x, \xi) d\xi dA \\
- T_0 \int_A \int_0^t p^{(1)}(x, t - \xi)[a_i \hat{p}_6(\gamma^{(2)})(x, \xi) + n b_j \hat{p}_5(s^{(2)})(x, \xi)] d\xi dA \\
- n(\tau^2 - \tau^0)T_0 \int_V \int_0^t \gamma^{(1)}(x, t - \xi) \frac{\partial \hat{p}_4}{\partial \xi} (\gamma^{(2)})(x, \xi) d\xi dV = S_{21}^{12}.
\end{align*}
\]

(139)