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Uniform stabilization of the telegraph equation with a support by fuzzy transform method

Rajib Ghosh[1,*], Sanchita Chowdhury[2], Ganesh Chandra Gorain[1], Samarjit Kar[3]

[1]Department of Mathematics, J. K. College, Purulia, West Bengal, 723101, India
[2]Barapalashon Girls' High School, Barapalashon, Burdwan, West Bengal, India
[3]Department of Mathematics, N. I. T, Durgapur, 713209, West Bengal, India
*Email: rajibghosh0509@gmail.com

ABSTRACT

We consider the vibrations of electrical waves or telecommunication signals. The uniform stabilization of such vibrations is directly established with an explicit form of exponential energy decay estimate. Using the fuzzy transform method, a closed form numerical scheme is constructed to support the stability result.

Keywords: telegraph equations, uniform stability, exponential energy decay estimate, Lyapunov function, fuzzy transform

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1. INTRODUCTION AND MATHEMATICAL FORMULATION

The telegraph equation arises from the propagation of electrical signals along a telegraph line. In 1850, the mathematical theory on this equation was first explored by Lord Kelvin, based on the idea of the signal decaying in underwater telegraph cables. At that time, William Thomson investigated the same with the help of Fourier's equation for heat conduction in a wire. On the other hand, Hodgkin and Huxley[1] applied this idea of signal transmission to different parts of neurons within neuron cell membranes.

In this study, we consider the vibrations of a clamped elastic string, mathematically governed by the standard telegraph equation

$$u_{xx}(x,t) = au_{tt}(x,t) + bu_t(x,t) + cu(x,t), \quad 0 < x < L, t > 0, \tag{1}$$

where the coefficients a,b,c are all constants.

The stabilization of the vibrations of a flexible structure is a problem in a dynamical system governed by partial differential equations. The most common class of vibrational stability is of the passive type, which uses a resistive device to absorb vibration energy. Though the vibrations of flexible structures are nonlinear in practice, linearized models are treated for analytical approach, simplicity and for concise results. The problem of energy decay estimates, in the context of wave equation, were studied by several authors (cf. Chen,[2,3] Gorain,[4,5] Komornik,[6] Shahruz,[7] Nandi et al.[8] and a list of references therein).

The boundary conditions are

$$u(0,t) = 0 \quad and \quad u(L,t) = 0, \quad t \geq 0 \tag{2}$$

Initially, this string is set to vibrate with

$$u(x,0) = f(x) \quad and \quad u_t(x,0) = g(x), \quad 0 \leq x \leq L. \tag{3}$$

The function $f(x)$ and $g(x)$ are assumed to be continuous upto second order partial derivatives over $[0,L]$, so that the solution $u(x,t)$ is continuously differentiable over $(0,L)\times(0,\infty)$.

The aim of the present paper is to study the result of uniform stability for the solution of the mathematical problem (1), subject to the boundary and initial conditions (2) – (3), by the means of an explicit form of exponential energy decay estimate. To get this result, we use a direct method of constructing a suitable Lyapunov function related to the energy functional of the system (1) – (3). This exponential result had been directly obtained by Gorain[9] for an inhomogeneous beam, Lagnese[10] for a wave equation in R^n and by Nandi et al.[11] for the vibrations of a solar panel.

The study of dynamical systems modeled by differential equations is sometimes incomplete or vague. Functional relationships connecting different parameters of a system truly characterise the whole set of the system behavior, to be compatible with our limited knowledge of the system. This idea leads to the Fuzzy Input Fuzzy Output (FIFO) system. In this paper, we would like to introduce and apply the technique of 'fuzzy transform' following the idea of Stepnicka[12] and Perfilieva et al.,[13] to the mathematical system (1) – (3) This technique is based on two transforms. One is direct fuzzy transform or, F-transform and other is inverse F-transform. Practically, numerical computation based on these transforms has gained importance due to its wide application to differential equations, especially on partial differential equations. On application of this method, a partial differential equation reduces to a set of algebraic equations. The treatment of these algebraic equations make it easier to obtain a numerical solution of the corresponding partial differential equation. There are many other numerical methods to handle such equations, but practically, this method is very helpful to verify the analytical result obtained through numerical computations. Another main interest of this work is to obtain an approximate closed-form numerical solution of the above system (1) – (3), by using the fuzzy transform method and then plotting the solution graphically for different values of the parameters, we can directly verify the analytical result of uniform stability.

2. ENERGY OF THE SYSTEM

We define the energy $E(u(t))$ of the system (1) – (3) at any instant t, by

$$E(u(t)) = \frac{1}{2}\int_0^L \left[au_t^2 + u_x^2 + cu^2\right]dx \quad for\ all \quad t \geq 0. \tag{4}$$

Now, differentiating (4) with respect to t, we have

$$\frac{dE}{dt} = \int_0^L [au_t u_{tt} + u_x u_{xt} + cuu_t]dx. \tag{5}$$

Using (1) in (5) and applying the boundary conditions in (2), we get

$$\frac{dE}{dt} = -b\int_0^L u_t^2 dx \leq 0 \quad \textit{for all} \quad t \geq 0. \tag{6}$$

It follows from (6) that the system (1) – (3) is energy dissipating. Integrating (6) with respect to t over $[0,t]$, the solution $u(x,t)$ satisfies the energy estimate

$$E(u(t)) - E(u(0)) = \int_0^L \int_0^t u_\tau^2 dx d\tau \quad \textit{for all} \quad t \geq 0, \tag{7}$$

where

$$E(u(0)) = \frac{1}{2}\int_0^L \left[a(g)^2 + (f_x)^2 + cf^2\right]dx. \tag{8}$$

In view of (7) and (8), we may conclude that if $f \in H_0^1(0,L) \quad \textit{and} \quad g \in L^2(0,L)$, where,

$$H_0^1(0,L) = \{\phi : \phi \in H^1(0,L), \phi(0) = \phi(L) = 0\}$$

is the subspace of the classical Sobolev space

$$H^1(0,L) = \{\phi : \phi \in L^2(0,L), \phi_x \in L^2(0,L)\}$$

of a real valued function of the order one, then

$$E(u(t)) \leq E(u(0)) < \infty \quad \textit{for} \quad t \geq 0. \tag{9}$$

3. UNIFORM STABILITY RESULT BY A DIRECT METHOD

Since the system (1) – (3) is not conserving and energy dissipating, it is natural to ask whether the solution of the system decays with time uniformly? For an affirmative answer to this question, we use a method that enables us to explicitly establish the uniform exponential energy decay estimate for the system.

Theorem: If $u(x,t)$ be the solution of the system (1) – (3), with $(f,g) \in H_0^1(0,L) \times L^2(0,L)$, then the solution $u(x,t) \longrightarrow 0$ exponentially as time $t \longrightarrow +\infty$, means, the energy functional E given by (4) satisfies

$$E(u(t)) \leq Ae^{-\gamma t}E(u(0)), \quad \textit{for} \quad t \geq 0, \tag{10}$$

for some reals $\gamma > 0$ and $A > 1$. The constants γ and A depend on the interval $[0,L]$ and eventually on the initial values $\{f,g\}$.

Proof: See the Appendix.

This result shows the uniform exponential stability of the system (1)-(3) by means of an exponential energy decay estimate. Hence, the solution of the system $u(x,t) \longrightarrow 0$ uniformly and exponentially as time $t \longrightarrow +\infty$ for every $(f,g) \in H_0^1(0,L) \times L^2(0,L)$.

We shall now verify the above result by a closed form numerical scheme using fuzzy transform or F-transform technique. To study the scheme, the basic ideas of F-transform is discussed briefly in the following.

4. F-TRANSFORMS FOR FUNCTIONS

This section recalls the method published by Perfilieva and Chaldeeva[13] and Stepnicka.[12] This technique is more numerical than linguistic, which is why it belongs to the area called numerical methods on the basis of fuzzy application models. An interval [a,b] of real numbers has been used as a common domain of all functions. Zadeh[14] introduced the concept of fuzzy sets via membership function as a mathematical means of describing vagueness in linguistics. In last two decades, many theoretical as well as numerical developments in fuzzy logic took place among the researchers of mathematical communities.

Let $x_i = a + b(i-1), \quad i = 1, 2, \ldots n$ be nodes on [a,b], where, $h = (b-a)/(n-1), n > 2$. We say that functions $A_1(x), \ldots, A_n(x)$ defined on [a,b] are basic functions, if each of them fulfills the following conditions:

(i) $A_i : [a,b] \longrightarrow [0,1], A_i(x_i) = 1$,
(ii) $A_i(x) = 0$ if $x \notin (x_{i-1}, x_{i+1})$, where, $x_0 = a, x_{n+1} = b$,
(iii) $A_i(x)$ is continuous,
(iv) $A_i(x)$ strictly increases on $[x_{i-1}, x_i]$ and strictly decreases on $[x_i, x_{i+1}]$,
(v) $\sum_{i=1}^{n} A_i(x) = 1$ for all $x \in [a,b]$,
(vi) $A_i(x_i - x) = A_i(x_i + x)$, for all $x \in [0,h], i = 2, \ldots n-1, n > 2$,
(vii) $A_{i+1}(x) = A_i(x-h)$, for all $x \in [a+h, b], i = 2, \ldots n-2, n > 2$.

In this case, we say that basic functions $A_1(x), \ldots, A_n(x)$ determine a uniform fuzzy partition of interval [a,b]. In other words, basic functions are the fuzzy sets determining a uniform fuzzy partition of real interval [a,b]. We have already mentioned that the technique of fuzzy transform is based on two transforms: one is the direct F transform and the other is the inverse F transform.

Let $f(x)$ be a continuous function on [a,b], determining a uniform fuzzy partition of [a,b]. If we set

$$F_i = \frac{\int_a^b f(x) A_i(x) dx}{\int_a^b A_i(x) dx}, \quad i = 1, 2, \ldots, n. \tag{11}$$

Then the n-tuple of real number $[F_1, F_2, \ldots, F_n]$ is called the direct F-transform of f with respect to the basic functions $A_1(x), A_2(x), \ldots, A_n(x)$. Each F_i is called a component of F-transform and the totality can be viewed as an aggregate representation of the function f. On the other hand, if $F_n[f] = [F_1, \ldots, F_n]$ be the F-transform of f with respect to basic function $A_1, \ldots, A_n$, then the function

$$f_n^F(x) = \sum_{i=1}^{n} A_i(x) F_i \tag{12}$$

is called the inverse F-transform.

5. CLOSED FORM NUMERICAL SCHEME

In this section, we study the application of F-transform in solving the above system of equations (1)–(3), that means, we wish to discuss the numerical method based on F-transform to the following initial-boundary value problem

$$\begin{cases} u_{xx} = a u_{tt} + b u_t + c u, & 0 < x < L, 0 < t < T \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & 0 \le x \le L \\ u(0,t) = 0, \quad u(L,t) = 0, & t \ge 0. \end{cases} \tag{13}$$

On application of F-transform, the first equation of (13) is transformed into the following algebraic equation

$$F^2[u_{xx}] = aF^2[u_{tt}] + bF^2[u_t] + c[u], \tag{14}$$

where, $F^2[u_{tt}]$, $F^2[u_{xx}]$, $F^2[u_t]$ are the matrices of the F-transform components of u_{tt}, u_{xx}, u_t given by

$$F^2[u_{tt}] = \begin{bmatrix} u_{tt}^{11} & u_{tt}^{12} & \bullet\bullet\bullet & u_{tt}^{1m} \\ u_{tt}^{21} & u_{tt}^{22} & \bullet\bullet\bullet & u_{tt}^{2m} \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ u_{tt}^{n1} & u_{tt}^{n2} & \bullet\bullet\bullet & u_{tt}^{nm} \end{bmatrix},$$

$$F^2[u_{xx}] = \begin{bmatrix} u_{xx}^{11} & u_{xx}^{12} & \bullet\bullet\bullet & u_{xx}^{1m} \\ u_{xx}^{21} & u_{xx}^{22} & \bullet\bullet\bullet & u_{xx}^{2m} \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ u_{xx}^{n1} & u_{xx}^{n2} & \bullet\bullet\bullet & u_{xx}^{nm} \end{bmatrix}$$

and

$$F^2[u_t] = \begin{bmatrix} u_t^{11} & u_t^{12} & \bullet\bullet\bullet & u_t^{1m} \\ u_t^{21} & u_t^{22} & \bullet\bullet\bullet & u_t^{2m} \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ u_t^{n1} & u_t^{n2} & \bullet\bullet\bullet & u_t^{nm} \end{bmatrix}.$$

To determine the matrices $F^2[u_{tt}], F^2[u_{xx}]$ and $F^2[u_t]$, we replace the partial derivatives in (14) by approximation as

$$u_{tt} \approx \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2},$$

$$u_{xx} \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2},$$

$$u_t \approx \frac{u(x,t+k) - u(x,t)}{k}.$$

Next, we can approximate u_{tt}^{ij} as follows:

$$\begin{aligned} u_{tt}^{ij} &= \frac{\int\int \frac{\partial^2 u}{\partial t^2}(x,t) A^i(x) B^j(t)dxdt}{\int\int A^i(x)B^j(t)dxdt} \\ &\approx \frac{\int\int \left[\frac{u(x,t+k)-2u(x,t)+u(x,t-k)}{k^2}\right] A^i(x)B^j(t)dxdt}{\int\int A^i(x)B^j(t)dxdt} \\ &= \frac{1}{k^2}\frac{\int\int u(x,t+k)A^i(x)B^j(t)dxdt}{\int\int A^i(x)B^j(t)dxdt} - \frac{2}{k^2}\frac{\int\int u(x,t)A^i(x)B^j(t)dxdt}{\int\int A^i(x)B^j(t)dxdt} \\ &\quad + \frac{1}{k^2}\frac{\int\int u(x,t-k)A^i(x)B^j(t)dxdt}{\int\int A^i(x)B^j(t)dxdt} \\ &= \frac{1}{k^2}[u^{i,j+1} - 2u^{ij} + u^{i,j-1}]. \end{aligned} \tag{15}$$

Similarly,

$$u_{xx}^{ij} = \frac{1}{h^2}[u^{i+1,j} - 2u^{ij} + u^{i-1,j}] \tag{16}$$

$$u_t^{ij} = \frac{1}{k}\left[u^{i,j+1} - u^{ij}\right]. \tag{17}$$

Using the relations (15), (16) and (17), we obtain to the following recursive equation

$$\frac{a}{k^2}\left[u^{i,j+1} - 2u^{ij} + u^{i,j-1}\right] - \frac{1}{h^2}\left[u^{i+1,j} - 2u^{ij} + u^{i-1,j}\right] + \frac{b}{k}\left[u^{i,j+1} - u^{ij}\right] + cu^{ij} = 0$$

or,

$$u^{i,j+1} = A\left[u^{i+1,j} + u^{i-1,j}\right] + Bu^{ij} - Cu^{i,j-1}, \tag{18}$$

for $i = 1, 2, \ldots, m-1$, $j = 0, 1, 2, \ldots, n-1$, where

$$A = \frac{k^2}{h^2(a+bk)}, \quad B = \frac{h^2(2a+bk) - k^2(2+ch^2)}{h^2(a+bk)}, \quad C = \frac{a}{(a+bk)}.$$

To solve the above recursive relation (18), we first put $j = 0$ in (18). Then we obtain

$$u^{i1} = A[u^{i+1,0} + u^{i-1,0}] + Bu^{i0} - Cu^{i,-1}, \quad i = 1, 2, \ldots m-1. \tag{19}$$

The unknown $u^{i,-1}$ for $i = 0, 1, \ldots, m$ occurring in recursive equation (19) can be obtained from the second initial condition $u_t(x, 0) = g(x)$ in (13), which leads to the following difference scheme.

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+k) - u(x, t-k)}{2k}. \tag{20}$$

In particular for $k = 1$, the above reduces to the recursive relation

$$u^{i,-1} = u^{i1} - 2kg(ih), \quad i = 1, 2, \ldots, m.$$

Substituting this value of $u^{i,-1}$ in equation (19), we can obtain $u^{i,1}$ as

$$u^{i1} = \frac{A}{1+C}[u^{i+1,0} + u^{i-1,0}] + \frac{B}{1+C}u^{i0} + 2\frac{C}{1+C}kg(ih), \tag{21}$$

for $i = 1, 2, \ldots, m-1$. Thus, we can obtain all the values of u^{ij} for $j = 1$ level, since $u^{i,0} = f(ih)$ are known for $i = 0,1,2,\ldots,m$, from the given function $f(x)$.

For second and higher order levels, we put $j = 1, 2, \ldots, n-1$, in the recursive relation (19), where

$$u^{0,j} = 0, \quad u^{m,j} = 0, \quad j = 0, 1, 2, \ldots, n$$

followed from the boundary conditions $u(0, t) = u(L, t) = 0$ in (13).

Applying the above computational scheme with different values of the parameters, the dynamical responses of the solution are shown in the following figures.

Figure 5.1 is obtained with parameters $a = 1$, $b = 1$, $c = 1$, $k = 0.05$, $h = 0.02$, $f(x) = sin(\pi*x)$, $g(x) = -0.02*x$ over the interval [0,5], while figure 5.2 is presented with parameters $a = 1.5$, $b = 2$, $c = -1.2$, $k = 0.05$, $h = 0.2$, $f(x) = x*\cos(\pi*x)/5, g(x) = 0.001*x$ over the interval [0,4.5] and figure 5.3 is illustrated with parameters $a = 1$, $b = 1.2$, $c = 1$, $k = 0.05$, $h = 0.1$, $f(x) = 0.5*\exp(0.1*x)*\sin(\pi*x), g(x) = 0.01*x$ over the interval [0,4].

After plotting the above numerical results through different graphs, we observe that the computational results obtained by F-transform method, satisfy the analytical results of the uniform exponential decay of solution. This is highly significant in the study of uniform stability of the telegraph equation.

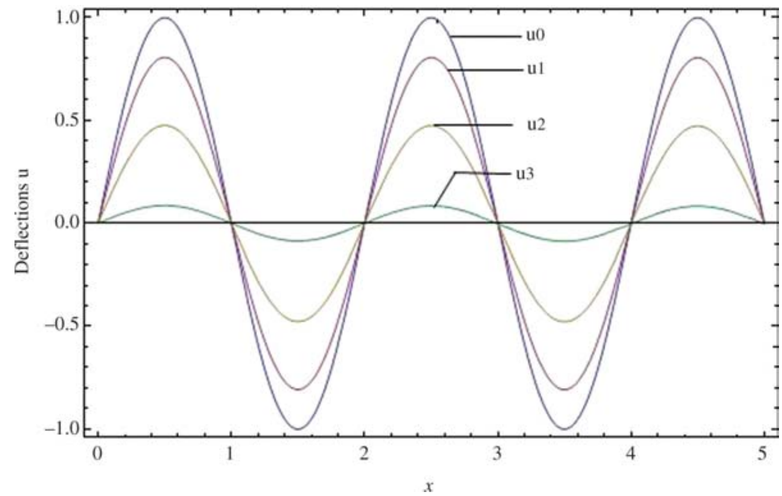

Figure 5.1. *Approximate deflections u(x,t) of the string for different* values of time. u_0, u_1, u_2, u_3 are the deflections of the string for the numerical value of $t = 0$, $t = 0.20$, $t = 0.35$, $t = 0.5$, respectively.

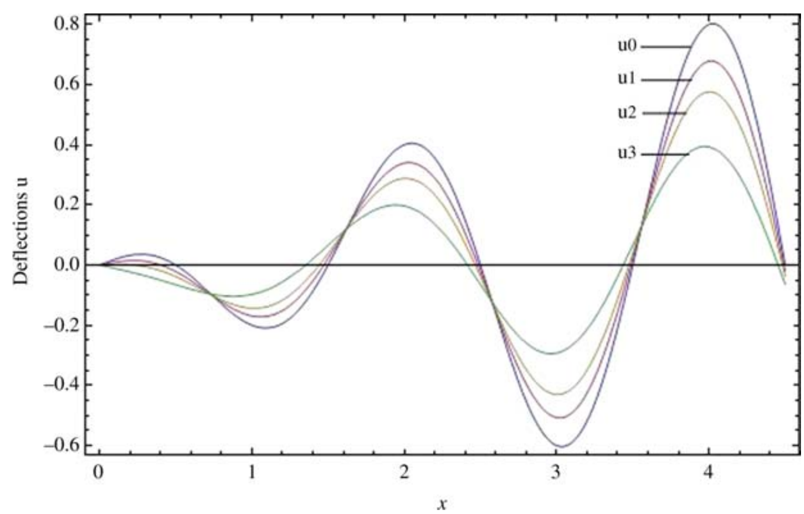

Figure 5.2. *Approximate deflections u(x,t) of the string for different* values of time. u_0, u_1, u_2, u_3 are the deflections of the string for the numerical value of $t = 0$, $t = 0.20$, $t = 0.35$, $t = 0.50$, respectively.

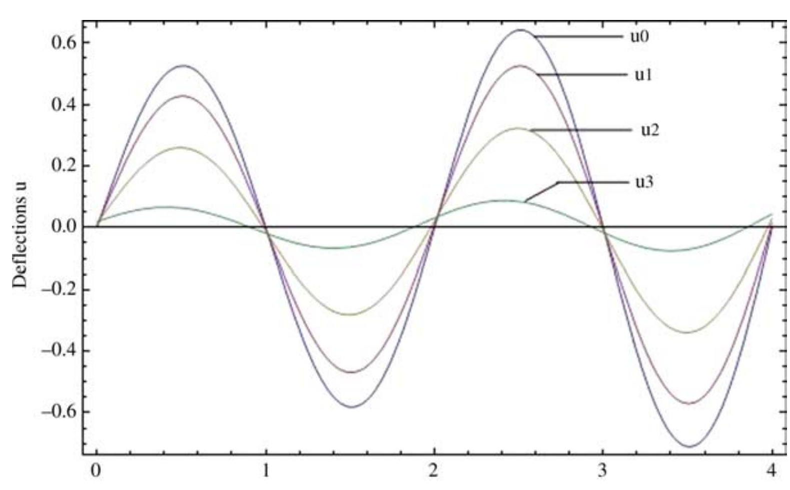

Figure 5.3. *Approximate deflections u(x,t) of the string for different* values of time. u_0, u_1, u_2, u_3 are the deflections of the string for the numerical value of $t = 0$, $t = 0.20$, $t = 0.35$, $t = 0.5$, respectively.

6. CONCLUSION

The present mathematical study deals with the uniform exponential stability result for the vibration of a telegraph equation, modeled by the linear differential equation (1). The uniform decay of solution is obtained by means of an exponential energy decay estimate. As the system is uniformly stable, it is controllable, in particular, from an arbitrary initial state to a desirable final state. Moreover, the result of uniform stability is verified by constructing a closed form numerical scheme, with the idea of fuzzy-transform method. The plotted graphs thus obtained verify the analytical results. Our discussion in this presentation covers the case of uniform stability of other structural vibrations of flexible structures, such as the vibration of rods, beams, plates etc.

Appendix Proof of theorem: To prove the theorem, we need the following inequalities as follows: For any real number $\alpha > 0$, we have Young's inequality (cf. Mitrinovic et al.[15])

$$|f.g| \leq \frac{1}{2}\left(\alpha|f|^2 + \frac{|g|^2}{\alpha}\right). \tag{22}$$

Moreover, we have the following Poincare type Scheeffer's inequality (cf. Mitrinovic et al.[15])

$$\int_0^L u^2 dx \leq \frac{L^2}{\pi^2}\int_0^L (u_x)^2 dx \tag{23}$$

because u satisfies the boundary conditions in (2). To establish the theorem, we proceed as in Komornik,[16] Gorain[4] and introduce an energy-like Lyapunov functional V, defined as

$$V(u(t)) = E(u(t)) + \varepsilon G(u(t)) \quad for \quad t \geq 0, \tag{24}$$

where $\varepsilon > 0$ is a small real number and

$$G(u(t)) = \int_0^L \left[auu_t + \frac{b}{2}u^2\right]dx. \tag{25}$$

Differentiating (25) with respect to t, we have

$$\frac{dG}{dt} = \int_0^L au_t^2 dx + \int_0^L auu_{tt} dx + \int_0^L buu_t dx \tag{26}$$

Using (1) and (2), we get

$$\frac{dG}{dt} = a\int_0^L u_t^2 dx - \int_0^L u_x^2 dx - c\int_0^L u^2 dx. \tag{27}$$

Using energy equation (4) in (27), we get

$$\frac{dG}{dt} = -2E(u(t)) + 2a\int_0^L u_t^2 dx. \tag{28}$$

Further using the inequalities (22) and (23), we can write,

$$\begin{aligned}
\left|\int_0^L uu_t dx\right| &= \frac{1}{a}\left|\int_0^L uau_t dx\right| \\
&\leq \frac{1}{2a}[\frac{\pi}{L}\int_0^L u^2 dx + \frac{aL}{\pi}\int_0^L u_t^2 dx] \\
&= \frac{1}{2a}\frac{L}{\pi}\int_0^L [u_x^2 + au_t^2]dx \\
&\leq \frac{L}{\pi a}E(u(t)) \\
&= \lambda_0 E(u(t)) \quad for \quad t \geq 0,
\end{aligned} \tag{29}$$

where

$$\lambda_0 = \frac{L}{\pi a}. \tag{30}$$

Also using (23), we can write

$$\begin{aligned}\int_0^L \frac{b}{2} u^2 dx &\le \frac{b}{2}\frac{L^2}{\pi^2}\int_0^L u_x^2 dx \\ &\le \lambda_1 E(u(t)) \text{ for } t \ge 0,\end{aligned} \tag{31}$$

where

$$\lambda_1 = \frac{bL^2}{\pi^2}. \tag{32}$$

Now, we can estimate (25) as

$$-\lambda_0 E(u(t)) \le G(u(t)) \le (\lambda_0 + \lambda_1)E(u(t)) \quad \text{for } t \ge 0$$

by virtue of (29) and (31). Hence, $V(u(t))$ as defined in (24) can be estimated as

$$(1 - \lambda_0\varepsilon)E(u(t)) \le V(u(t)) \le (1 + (\lambda_0 + \lambda_1)\varepsilon)E(u(t)) \quad \text{for } t \ge 0, \tag{33}$$

where we choose $\varepsilon < \frac{1}{\lambda_0}$ *so that* $V(u(t)) \ge 0$ *for* $t \ge 0$.

Now, taking the time derivative of (24) and applying the results (6) and (28), we get

$$\begin{aligned}\frac{dV}{dt} &= -2\varepsilon E(u(t)) - (b - 2a\varepsilon)\int_0^L u_t^2 dx \\ &\le -\frac{2\varepsilon}{1 + (\lambda_0 + \lambda_1)\varepsilon} V - (b - 2a\varepsilon)\int_0^L u_t^2 dx\end{aligned} \tag{34}$$

by virtue of relation (33). Since, ε is small, we assume that

$$0 < \varepsilon < \varepsilon_0 = min\left\{\frac{1}{\lambda_0}, \frac{b}{2a}\right\}. \tag{35}$$

Hence, (34) leads to the differential inequality

$$\frac{dV}{dt} + \gamma V \le 0 \, \text{for } t \ge 0, \tag{36}$$

where,

$$\gamma = \frac{2\varepsilon}{1 + (\lambda_0 + \lambda_1)\varepsilon}. \tag{37}$$

Multiplying (36) by $e^{\gamma t}$ and integrating over $[0,t]$, we get

$$V(u(t)) \le e^{-\gamma t} V(u(0)). \tag{38}$$

By using the inequality (32) in (38), we finally obtain the result

$$E(u(t)) \le Ae^{-\gamma t}E(u(0)), \text{ for } t \ge 0,$$

where,

$$A = \frac{1 + (\lambda_0 + \lambda_1)\varepsilon}{1 - \lambda_0\varepsilon} > 1. \tag{39}$$

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