QSCIENCE connect
A Qatar Foundation Academic Journal

Research paper

Exp-function method using modified Riemann-Liouville derivative for Burger's equations of fractional-order

Qazi Mahmood Ul Hassan, Syed Tauseef Mohyud-Din*

HITEC University Taxila Cantt, Pakistan
*Email: syedtauseefs@hotmail.com

ABSTRACT
This paper shows the combination of an efficient transformation and Exp-function method, to construct generalized solitary wave solutions of the nonlinear Burger's equations of fractional-order. Computational work and subsequent numerical results re-confirm the efficiency of the proposed algorithm. It is observed that the suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.

Keywords: Burger's equations, fractional calculus, exp-function method, modified Riemann-Liouville derivative

PACS: 02.30 Jr, 02.00.00

http://dx.doi.org/10.5339/connect.2013.19

Submitted: 30 April 2013
Accepted: 6 June 2013

QSCIENCE.com
Member of Qatar Foundation

Cite this article as: Ul Hassan QM, Mohyud-Din ST. Exp-function method using modified Riemann-Liouville derivative for Burger's equations of fractional-order, *QScience Connect* 2013:19
http://dx.doi.org/10.5339/connect.2013.19

1. INTRODUCTION

The subject of factional calculus[1,2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential Equations (NFDEs) have gained much interest due to the exact description of nonlinear phenomena of many real-time problems. Fractional calculus is also considered as a novel topic,[3,4] it has recently gained considerable popularity and importance. Fractional calculus has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous and diverse fields of science and engineering. Some of the present-day applications of fractional models[5–8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, biosciences, economics, geology, astrophysics, probability and statistics, chemical physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations.[1–8] Recently, He and Wu,[9] developed a very efficient technique called Exp-function method, for solving various nonlinear physical problems. The literature reveals that Exp-function method has been applied to a wide range of differential equations and is highly reliable. The Exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. Subsequent works have shown the complete reliability and efficiency of this algorithm. He et al.[10–11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din[12–15] extended the same for nonlinear physical problems, including higher-order BVPs; Oziz[16] tried this novel approach for Fisher's equation; Wu et al.[17,18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu[19] for MBBN equations; Zhang[20] for high-dimensional nonlinear evolutions; Zhu[21,22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov[23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani[24] for an explicit and numerical solutions of the fractional KdV equation.

The motivation for this paper is the development of an efficient combination comprising an efficient transformation, Exp-function method using Jumarie's derivative approach,[25–28] and its subsequent application to construct generalized solitary wave solutions of the nonlinear Burger's Equations of fractional-order.[29–30] It is worth mentioning that Ebaid [31] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation. It is to be noted that fractional Burger's equations[32–36] describe the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the memory effect of the wall friction through the boundary layer. The same form can be found in other systems such as shallow-water waves and waves in bubbly liquids. Generally, a boundary layer will give rise to memory effects in the form of this fractional derivative. Moreover, such equations are of utmost importance in mathematical physics and engineering sciences. Hence these appear quite often in a number of scientific models, including fluid mechanics, astrophysics, solid state physics, plasma physics, chemical kinematics, chemical chemistry, optical fiber and geochemistry.[11,13,34–38]

Theorem 1.1:[31] Suppose that $u^{(r)}$ and u^{γ} are the highest order linear term and the highest order nonlinear term of a nonlinear ODE, respectively, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}$, leads to $c = d$ and $p = q$, $\forall r, s, \Omega, \lambda \geq 1$.

Theorem 1.2:[31] Suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are the highest order linear term and the highest order nonlinear term of a nonlinear ODE, respectively, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s, k \geq 1$.

Theorem 1.3:[31] Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}$ are the highest order linear term and the highest order nonlinear term of a nonlinear ODE, respectively, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s \geq 1, \forall \Omega \geq 2$.

Theorem 1.4:[31] Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}u^{\lambda}$ are the highest order linear term and the highest order nonlinear term of a nonlinear ODE, respectively, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q$, $\forall r, s, \Omega, \lambda \geq 1$.

2. JUMARIE'S FRACTIONAL DERIVATIVE

Jumarie's fractional derivative is a mo1dified Riemann-Liouville derivative defined as;

$$D_t^{\alpha} f(x) = \left\{ \begin{array}{l} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-t)^{-\alpha-1} \big(f(t) - f(0)\big) dt, \alpha \leq 0, \\ \frac{1}{\Gamma(-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} \big(f(t) - f(0)\big) dt, 0 \leq \alpha \leq 1 \\ \left[f^{\alpha-n}(x)^n\right]^n, n \leq \alpha \leq n+1, n \geq 1 \end{array} \right\} \tag{1}$$

Where $f : R \to R, x \to f(x)$ denotes a continuous (but not necessarily differentiable) function.

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative are summarised in the references[25–28].

$$D_x^{\alpha} c = 0, \alpha \geq 0, \ c = \text{constant} \tag{2}$$

$$D_x^{\alpha}\left[cf(x)\right] = cD_x^{\alpha} f(x) \alpha \geq 0, \ c = \text{constant} \tag{3}$$

$$D_x^{\alpha} x^{\beta} = \frac{\Gamma\big(1+\beta\big)}{\Gamma\big(1+\beta-\alpha\big)} x^{\beta-\alpha}, \beta \geq \alpha \geq 0. \tag{4}$$

$$D_x^{\alpha}\left[f(x)g(x)\right] = \left[D_x^{\alpha} f(x) g(x) + f(x)\left[D_x^{\alpha} g(x)\right]\right]. \tag{5}$$

$$D_x^{\alpha} f(x(t)) = f_x^{'}(x) . x^{\alpha}(t). \tag{6}$$

3. EXP-FUNCTION METHOD[11,14,38,39]

We consider the general nonlinear FPDE of the type

$$P(u, u_t, u_x, u_{xx} u_{xxx}, \ldots, D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u, \ldots) = 0, \quad 0 < \alpha \leq 1, \tag{7}$$

where $D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx respectively.

Using a transformation[39]

$$\eta = kx + my + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0, \quad k, \omega, m, \eta_0 \text{ are all constants with } k, \omega, \neq 0 \tag{8}$$

we can rewrite equation (7) in the following nonlinear ODE;

$$Q(u, u', u'', u''', u^{iv}) = 0, \tag{9}$$

where the prime denotes derivative with respect to η.

According to Exp-function method, we assume that the wave solution can be expressed in the following form

$$u(\eta) = \frac{\sum_{n-c}^{d} a_n \exp\left[n\eta\right]}{\sum_{m-p}^{q} b_m \exp\left[m\eta\right]} \tag{10}$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite Eq. (4) in the following equivalent form

$$u(\eta) = \frac{a_c \exp(c\eta) + \ldots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \ldots + b_{-q} \exp(-q\eta)}. \tag{11}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p by using[31],

$$p = c, q = d. \tag{12}$$

4. NUMERICAL APPLICATIONS

In this section, we apply Exp-function method to construct generalized solitary solutions for Burger's Equations of fractional-order. The numerical results are very encouraging.

EXAMPLE 4.1: Consider the following Burger's Equation of fractional order

$$D_t^{\alpha} u + u u_x = u_{xx}, \quad 0 < \alpha \leq 1. \tag{13}$$

Using (8) equation (13) can be converted to an ordinary differential equation

$$\omega u' + k u u' = k^2 u'', \tag{14}$$

where the prime denotes the derivative with respect to η. The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and$p$, by using[31],

$$p = c, q = d. \tag{15}$$

CASE 4.1.I: We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of $c$and d. For simplicity, we set $p = c = 1$and
$q = d = 1$ equation (11) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{16}$$

Substituting equation (16) into equation (14), we have

$$\frac{1}{A}\left[\begin{matrix} c_4 \exp(4\eta) = c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) \end{matrix}\right] = 0, \tag{17}$$

where $A = \left(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)\right)^4$,$c_i$ are constants obtained by Maple 16. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\{c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\}. \tag{18}$$

Solution of (12) will yield

$$\left\{a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_0 = \frac{a_1 b_0}{b_1}, a_1 = a_1, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1\right\}. \tag{19}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$of equation (13)(Figure 4.1)

$$\left\{\frac{\frac{b_{-1}(2k^2-\omega)e^{-\frac{xk+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{k} - \frac{b_1(2k^2+\omega)e^{\frac{xk+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{k}}{b_{-1}e^{-\frac{xk+\omega t^{\alpha}}{\Gamma(1+\alpha)}} + b_1 e^{\frac{xk+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}\right\}. \tag{20}$$

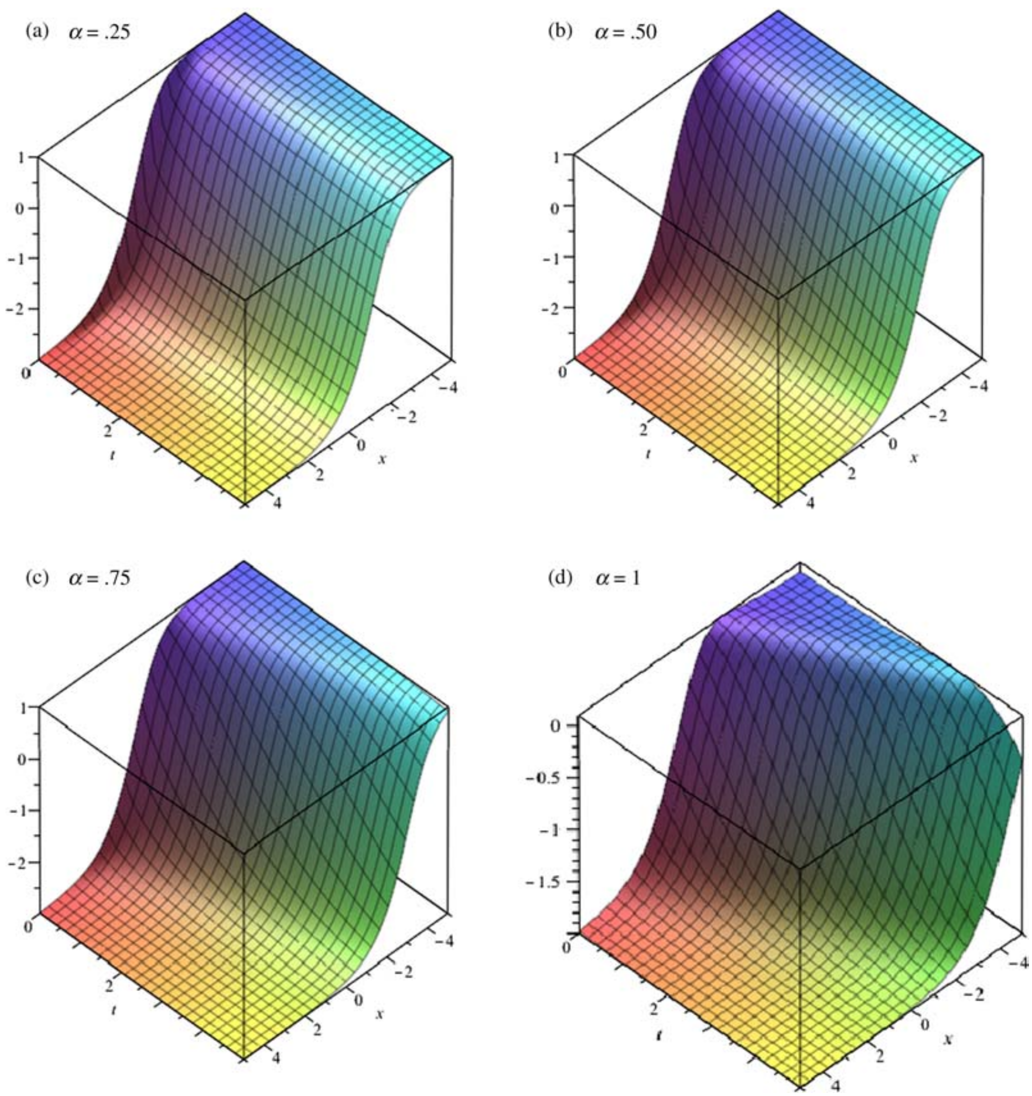

Figure 4.1. (a), (b), (c) and (d). Soliton solutions of equation (13) for $a_0 = b_1 = b_{-1} = \omega = 1$ and $k = 1$.

CASE 4.1.II: If $p = c = 2$ and $q = d = 1$, then trial solution equation (13) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta]}. \tag{21}$$

Proceeding as before, we obtain

$$\left\{a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_0 = \frac{a_1 b_0}{b_1}, a_1 = a_1, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1\right\}. \tag{22}$$

Hence we get the generalized solitary wave solution of Equation (13) for $\alpha = 1$ as follows

$$\left\{\frac{\frac{b_{-1}(2k^2-\omega)e^{-(xk+\omega t)}}{k} - \frac{b_1(2k^2+\omega)e^{xk+\omega t}}{k}}{b_{-1}e^{-(xk+\omega t)} + b_1 e^{xk+\omega t}}\right\}. \tag{23}$$

In both cases, for different choices of c, p, d and q, we get the same soliton solutions, which clearly illustrates that the final solution does not strongly depend upon these parameters.

Example 4.2: Consider the following Burger's equation of fractional order

$$D_t^\alpha u + \beta u_{xx} + 2\beta u u_x + \delta(u_x + u_y) = 0, \quad 0 < \alpha \leq 1. \tag{24}$$

Using (8) equation (24) can be converted to an ordinary differential equation

$$\omega u' + \beta k^2 u'' + 2\beta u u' + \delta(k+m)u' = 0, \tag{25}$$

where the prime denotes the derivative with respect to η. The solution of the equation (24) can be expressed in the form, equation (11). To determine the value of c and$p$, by using[31],

$$p = c, q = d. \tag{26}$$

Case 4.2.I: We can freely choose the values of c and d, but we illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (11) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{27}$$

Substituting equation (27) into equation (25), we have

$$\frac{1}{A}\left[\begin{array}{c} c_4 \exp(4\eta) = c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) \end{array}\right] = 0, \tag{28}$$

where $A = \left(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)\right)^4$, c_i are constants obtained by Maple 16. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\{c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\}. \tag{29}$$

Solution of (29) will yield

$$\left\{\begin{array}{l} a_0 = 0, a_{-1} = -\frac{1}{2}\frac{b_{-1}(\omega + 2\beta k^2 + \delta k + \delta m)}{\beta k}, \\ a_1 = \frac{1}{2}\frac{b_1(-\omega + 2\beta k^2 - \delta k - \delta m)}{\beta k}, b_0 = 0, \\ a_1 = a_1, b_{-1} = b_{-1}, b_1 = b_1 \end{array}\right\}. \tag{30}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (24) (Figure 4.2)

$$\left\{\frac{-\frac{1}{2}\frac{b_{-1}(\omega + 2\beta k^2 + \delta k + \delta m)}{\beta k} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{1}{2}\frac{b_1(-\omega + 2\beta k^2 - \delta k - \delta m)}{\beta k} e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}}}{b_{-1} e^{-\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}} + b_1 e^{\frac{xk + \omega t^\alpha}{\Gamma(1+\alpha)}}}\right\} \tag{31}$$

Case 4.2.II: If $p = c = 2$ and $q = d = 1$, then trial solution equation (24) reduces to

$$u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta]}. \tag{32}$$

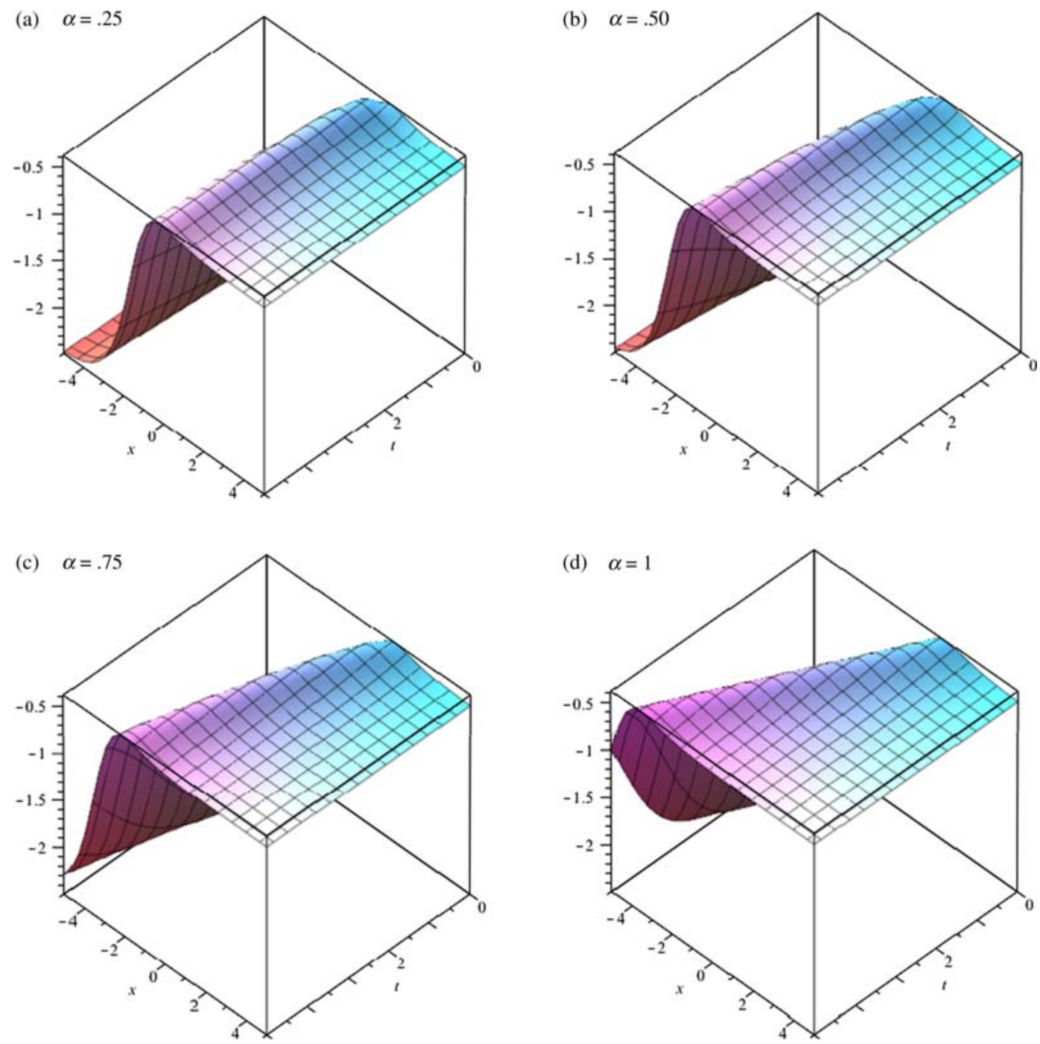

Figure 4.2. (a), (b), (c) and (d). Soliton solutions of equation (24) for $b_1 = b_{-1} = \omega == \beta = \delta = m = 1$ and $k = 1$.

Proceeding as before, we obtain

$$\left\{\begin{array}{l} a_{-1} = -\frac{1}{2}\frac{b_{-1}(\omega+2\beta k^2+\delta k+\delta m)}{\beta k}, \\ a_1 = \frac{1}{2}\frac{b_1(-\omega+2\beta k^2-\delta k-\delta m)}{\beta k}, \\ a_0 = 0, a_1 = a_1, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1 \end{array}\right\}. \tag{33}$$

Hence we get the generalized solitary wave solution of equation (24) for $\alpha = 1$ as follows

$$\left\{\frac{-\frac{1}{2}\frac{b_{-1}(\omega+2\beta k^2+\delta k+\delta m)}{\beta k}e^{-(xk+\omega t)} + \frac{1}{2}\frac{b_1(-\omega+2\beta k^2-\delta k-\delta m)}{\beta k}e^{xk+\omega t}}{b_{-1}e^{-(xk+\omega t)} + b_1 e^{xk+\omega t}}\right\} \tag{34}$$

In both cases, for different choices of c, p, d and q, and, we get the same soliton solutions which clearly illustrates that final solution does not strongly depends upon these parameters.

5. CONCLUSION

In this paper, we applied Exp-function method to construct generalized solitary solutions of the nonlinear fractional order Burger's equations. It is observed that the Exp-function method is very convenient to apply, and is very useful for finding solutions to a wide class of nonlinear problems.

REFERENCES

[1] Hilfer R, ed. *Applications of Fractional Calculus in Physics*. World Scientific; 2000.
[2] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam, The Netherlands: Elsevier; 2006:p.204.
[3] Miller KS, Ross B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York, NY, USA: Wiley-Interscience, John Wiley & Sons; 1993.
[4] Podlubny I. *Fractional Differential Equations. Mathematics in Science and Engineering*. San Diego, California, USA: Academic Press; 1999:p.198.
[5] Shawagfeh NT. Analytical approximate solutions for nonlinear fractional differential equations. *Appl Math Comput*. 2002;131(2-3):517–529.
[6] Saha Ray S, Bera RK. An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method. *Appl Math Comput*. 2005;167(1):561–571.
[7] He JH. Some applications of nonlinear fractional differential equations and their approximations. *Bull Sci Tech*. 1999;15(2):86–90.
[8] Yıldırım A, Mohyud-Din ST, Sarıaydın S. Numerical comparison for the solutions of anharmonic vibration of fractionally damped nano-sized oscillator. *J King Saud University Sci*. 2011;23(2):205–209.
[9] He J-H, Wu X-H. Exp-function method for nonlinear wave equations. *Chaos Soliton Fract*. 2006;30(3):700–708.
[10] HE J-H. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. *Int J Mod Phys B*. 2008;22(21):3487–3578.
[11] He J-H, Abdou MA. New periodic solutions for nonlinear evolution equations using Exp-function method. *Chaos Soliton Fract*. 2007;34(5):1421–1429.
[12] Mohyud-Din ST, Noor MA, Waheed A. Exp-function method for generalized traveling solutions of good Boussinesq equations. *J Appl Math Comput*. 2009;30(1-2):439–445.
[13] Mohyud-Din ST, Noor MA, Noor KI. Some Relatively New Techniques for Nonlinear Problems. *Math Probl Eng*. 2009;2009:1–26.
[14] Noor MA, Mohyud-Din ST, Waheed A. Exp-function method for solving Kuramoto-Sivashinsky and Boussinesq equations. *J Appl Math Comp*. 2009;29(1-2):1–13.
[15] Noor MA, Mohyud-Din ST, Waheed A. Exp-function Method for Generalized Traveling Solutions of Master Partial Differential Equation. *Acta Applicandae Mathematicae*. 2008;104(2):131–137.
[16] Öziş T, Köroğlu C. A novel approach for solving the Fisher equation using Exp-function method. *Phys Lett A*. 2008;372(21):3836–3840.
[17] (Benn)Wu X-H, He J-H. EXP-function method and its application to nonlinear equations. *Chaos Soliton Fract*. 2008;38(3):903–910.
[18] Wu X-H (Benn), He J-H. Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method. *Comput Math Appl*. 2007;54(7-8):966–986.
[19] Yusufoğlu E. New solitonary solutions for the MBBM equations using Exp-function method. *Phys Lett A*. 2008;372(4):442–446.
[20] Zhang S. Application of Exp-function method to high-dimensional nonlinear evolution equation. *Chaos Soliton Fract*. 2008;38(1):270–276.
[21] Zhu SD. Exp-function Method for the Hybrid-Lattice System. *Int J Nonlin Sci Num*. 2007;8(3).
[22] Zhu S-D. Exp-function Method for the Discrete mKdV Lattice. *Int J Nonlin Sci Num*. 2007;8(3).
[23] Kudryashov NA. Exact soliton solutions of the generalized evolution equation of wave dynamics. *J Appl Math Mech*. 1988;52(3):361–365.
[24] Momani S. An explicit and numerical solutions of the fractional KdV equation. *Math Comput Simulat*. 2005;70(2):110–118.
[25] He JH, Li ZB. Fractional complex transform for fractional differential equations. *Math Comput Appl*. 2010;15(5):970–973.
[26] Li Z-B. An Extended Fractional Complex Transform. *Int J Non Sci Num Simulat*. 2010;11(Supplement).
[27] He J-H, Li Z-B. Converting fractional differential equations into partial differential equations. *Therm Sci*. 2012;16(2):331–334.
[28] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. *Comput Math Appl*. 2006;51(9-10):1367–1376.
[29] Momani S. Non-perturbative analytical solutions of the space- and time-fractional Burgers equations. *Chaos Soliton Fract*. 2006;28(4):930–937.
[30] Wang Q. Homotopy perturbation method for fractional KdV-Burgers equation. *Chaos Soliton Fract*. 2008;35(5):843–850.
[31] Ebaid A. An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms. *J Math Anal Appl*. 2012;392(1):1–5.
[32] Ma WX. Complexiton solutions to the Korteweg-de Vries equation. *Phys Lett A*. 2002;301(1-2):35–44.
[33] Ma WX, Fuchssteiner B. Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. *Int J Nonlin Mech*. 1996;31(3):329–338.
[34] Ma W-X, You Y. Solving the Korteweg-de-Vries equation by its bilinear form: Wronskian solutions. *Tran American Math Soc*. 2005;357(05):1753–1779.
[35] Ma WX, You Y. Rational solutions of the Toda lattice equation in Casoratian form. *Chaos Soliton Fract*. 2004;22(2):395–406.
[36] Ma W-X, Wu H, He J. Partial differential equations possessing Frobenius integrable decompositions. *Phys Lett A*. 2007;364(1):29–32.
[37] Mohyud-Din ST. Solution of nonlinear differential equations by Exp-function method. *World Appl Sci J*. 2009;7:116–147.
[38] Abdou MA, Soliman AA, El-Basyony ST. New application of Exp-function method for improved Boussinesq equation. *Phys Lett A*. 2007;369(5-6):469–475.
[39] Zheng B. G'/G- Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics. *Commun Theor Phys*. 2012;58(5):623–630.